Math 16B
Kouba
Why is area an antiderivative?

Let \( A(x) = \int_a^x f(t) \, dt \) represent the area of the region under the graph of \( y = f(t) \) and above the \( t \)-axis from \( t = a \) to \( t = x \). Note that

\[
A(a) = \int_a^a f(t) \, dt = 0 \quad \text{and} \quad A(b) = \int_a^b f(t) \, dt.
\]

Find the derivative of \( A(x) \):

\[
A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{\text{"area of shaded strip, } x \text{ to } x+h\text{"}}{h}
\]

\[
= \lim_{h \to 0} \frac{\text{"area of rectangle formed by } t\text{"}}{h}
\]
\[
\lim_\limits{h \to 0} \frac{(\text{base})(\text{height})}{h} = \lim_\limits{h \to 0} \frac{h \cdot f(\varepsilon)}{h} = \lim_\limits{h \to 0} f(\varepsilon) = f(x) \quad \text{Thus, } A'(x) = f(x)
\]

so \(A(x)\) is an antiderivative for \(f(x)\). Let

\[
\int f(x) \, dx = F(x) + C
\]

be the most general antiderivative for \(f(x)\). It follows that

\[
A(x) = F(x) + C
\]

and \(A(a) = F(a) + C \to 0 = F(a) + C \to C = -F(a), \quad A(b) = F(b) + C = F(b) - F(a), \) i.e.,

\[
\int_a^b f(t) \, dt = F(b) - F(a) \quad \text{or}
\]

\[
\int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a)
\]

(Fundamental Theorem of Calculus)