Math 16B (Winter 2021)
Kouba
Exam 1

Printing and signing your name below is a verification that no other person assisted you in the completion of this Exam.

PRINT your name ______________________ SIGN your name KEY

Show clear, organized supporting work for your answers. Correct answers without supporting work may not receive full credit. Use of unapproved shortcuts may not receive full credit. There are 7 pages. You must submit exactly 7 pages to Gradescope.

1.) (3 pts. each) Determine whether each statement is true (T) or false (F). Then circle the appropriate response.

a.) $\dfrac{1}{x-y} = \dfrac{1}{x} - \dfrac{1}{y}$ T F

b.) $(\ln x)^m = m \ln x$ T F

c.) $\sqrt{x \cdot y} = \sqrt{x} \cdot \sqrt{y}$ T F

d.) $\dfrac{\ln x}{\ln y} = \ln x - \ln y$ T F

2.) (10 pts.) Solve the following equation for $t$ : $e^{2t} - 6 = e^t$ $\longrightarrow$

$e^{2t} - e^t - 6 = 0 \rightarrow (e^t)^2 - (e^t) - 6 = 0 \rightarrow$

$(e^t - 3)(e^t + 2) = 0 \rightarrow \cancel{e^t = -2}$ (NO)

or $e^t = 3 \rightarrow \ln e^t = \ln 3 \rightarrow$

$t = \ln 3$

3.) Let $y = x^2 \ln x$ .

a.) (6 pts.) Solve $y' = 0$ for $x$ . $\xrightarrow{D}$ $y' = x^2 \cdot \frac{1}{x} + 2x \cdot \ln x$

$= x + 2x \ln x = x(1 + 2\ln x) = 0 \rightarrow$

$\cancel{x = 0}$ (NO) or $\ln x = -\frac{1}{2}$ or $x = e^{-1/2}$

b.) (6 pts.) Solve $y'' = 0$ for $x$ .

$\xrightarrow{D}$ $y'' = x \cdot 2 \cdot \frac{1}{x} + (1)(1 + 2\ln x)$

$= 2 + 1 + 2\ln x = 3 + 2\ln x = 0 \rightarrow$

$\ln x = -3/2 \rightarrow x = e^{-3/2}$

4.) (10 pts.) You love bubblegum and you are chewing on a large piece of it. The sugar in your bubblegum has a half-life of 2 minutes. After 5 minutes your bubblegum has 20 grams of sugar. What was the original amount of sugar in your bubblegum ?

Let $A$: grams of sugar at time $t$ min.;
assume $A = Ce^{kt}$ and $\frac{1}{2}$-life: $t=2$, $A = \frac{1}{2}C$
where $C$ is initial amount $\rightarrow$

$\frac{1}{2}\cancel{C} = \cancel{C}e^{2k} \rightarrow \ln(1/2) = \ln e^{2k} = 2k \rightarrow k = \frac{1}{2}\ln\left(\frac{1}{2}\right) \rightarrow$

$A = Ce^{\left(\frac{1}{2}\ln\left(\frac{1}{2}\right)\right)t}$; then $t = 5$, $A = 20 \rightarrow$

$20 = Ce^{\left(\frac{1}{2}\ln\left(\frac{1}{2}\right)\right)(5)} = Ce^{\frac{5}{2}\ln\left(\frac{1}{2}\right)} \rightarrow$

$C = \frac{20}{e^{5/2\ln\left(\frac{1}{2}\right)}} \approx 113.1$ gms.

5.) A bowling ball is dropped (initial velocity is 0 ft./sec.) from a tall building 1600 feet high. Assume that the acceleration due to gravity is $-32$ ft./sec.$^2$

a.) (5 pts.) Clearly DERIVE formulas for the velocity and height (above the ground) of the doomed bowling ball.

$s'' = -32 \xrightarrow{A.D.} s' = -32t + c$

$(t = 0,\ s' = 0 \rightarrow 0 = -32(0) + c \rightarrow c = 0) \rightarrow$

velocity: $\boxed{s'(t) = -32t}$ $\xrightarrow{A.D.}$

$s = -16t^2 + c$ $(t = 0,\ s = 1600 \rightarrow$

$1600 = -16(0)^2 + c \rightarrow c = 1600) \rightarrow$

height: $\boxed{s(t) = -16t^2 + 1600}$

b.) (2 pts.) In how many seconds will the bowling ball strike the ground ?

strike ground: $s(t) = 0 \rightarrow$

$-16t^2 + 1600 = 0 \rightarrow 16t^2 = 1600 \rightarrow$

$t^2 = 100 \rightarrow t = 10$ sec.

c.) (2 pts.) What is the bowling ball's velocity as it strikes the ground ?

$s'(10) = -32(10) = -320$ ft./sec.

6.) (10 pts.) You invest \$500 in an account earning an annual interest rate of $r$, and your investment grows to \$2000 in 10 years. If interest is compounded monthly, what is the annual interest rate $r$ ?

Discrete: $A = P\left(1+\frac{r}{n}\right)^{nt} \longrightarrow$

$$2000 = 500\left(1+\frac{r}{12}\right)^{12(10)} \longrightarrow$$

$$4 = \left(1+\frac{r}{12}\right)^{120} \longrightarrow 4^{1/120} = \left(1+\frac{r}{12}\right)^{\cancel{120}\cdot\frac{1}{\cancel{120}}}$$

$$\longrightarrow 4^{1/120} = 1+\frac{r}{12} \longrightarrow \frac{r}{12} = 4^{1/120}-1$$

$$\longrightarrow r = 12\left(4^{1/120}-1\right) \approx 0.1394$$

$$\longrightarrow r \approx 13.94\%$$

7.) (10 pts.) Use implicit differentiation to determine $y' = \frac{dy}{dx}$ for $xy = e^{y^2} + 3^x$.

$$\xrightarrow{D} xY' + (1)Y = e^{Y^2}\cdot 2YY' + 3^X \ln 3$$

$$\longrightarrow xY' - 2Ye^{Y^2}Y' = 3^X \ln 3 - Y$$

$$\longrightarrow Y'\left(X - 2Ye^{Y^2}\right) = 3^X \ln 3 - Y$$

$$\longrightarrow Y' = \frac{3^X \ln 3 - Y}{X - 2Ye^{Y^2}}$$

8.) You invest in a technology stock and the value of your investment at time $t$ weeks is given by $V = 500(t+1)e^{(-1/5)t}$ dollars.

a.) (2 pts.) What is the initial value of your investment ?

$t=0: \; V = 500e^{0} = 500(1) = \$500$

b.) (2 pts.) What is the value of your investment when $t = 10$ weeks ?

$t=10: \; V = 500(11)e^{-2} \approx \$744.34$

c.) (6 pts.) What will be the MAXIMUM value of your investment and when will it occur ?

$$V' = 500(t+1)e^{-\frac{1}{5}t}\left(\frac{-1}{5}\right) + 500e^{-\frac{1}{5}t}$$
$$= 500e^{-\frac{1}{5}t}\left[-\frac{1}{5}t + \frac{-1}{5} + 1\right]$$
$$= 500e^{-\frac{1}{5}t}\left[\frac{1}{5}(4-t)\right] = 0$$

$+ \quad 0 \quad - \qquad V'$

$t = 4$ mo.

MAX $\quad V = 500(5)e^{-4/5} \approx \$1123.32$

9.) Determine the following FOUR antiderivatives (indefinite integrals).

a.) (8 pts) $\int (x^{2/3} + 7x^{-3} + 1)\ dx$

$$= \frac{3}{5} x^{5/3} + 7 \cdot \frac{1}{-2} x^{-2} + x + C$$

b.) (8 pts.) $\int \frac{x^3 - x^2 + 1}{x^2}\ dx$

$$= \int \left[ \frac{x^3}{x^2} - \frac{x^2}{x^2} + \frac{1}{x^2} \right] dx$$

$$= \int (x - 1 + x^{-2})\, dx = \frac{1}{2} x^2 - x + \frac{1}{-1} x^{-1} + C$$

c.) (8 pts.) $\int (x+1)(x^2+2x)^5\ dx$

$$\left( \text{Let } u = x^2 + 2x \xrightarrow{D} \right.$$

$$du = (2x+2)\, dx = 2(x+1)\, dx \rightarrow$$

$$\left. \frac{1}{2} du = (x+1)\, dx \right)$$

$$= \frac{1}{2} \int u^5\, du = \frac{1}{2} \cdot \frac{1}{6} u^6 + C = \frac{1}{12} (x^2 + 2x)^6 + C$$

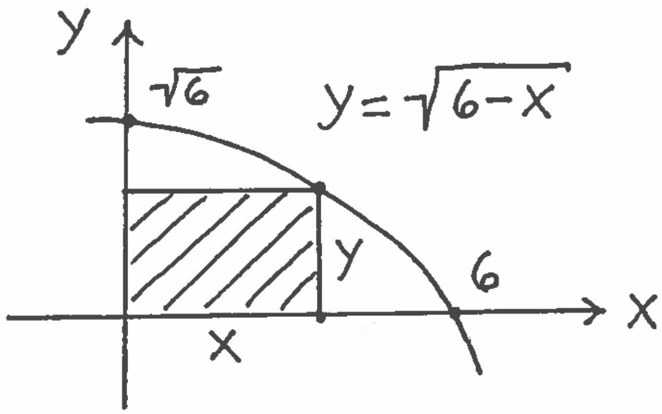
d.) (8 pts.) $\int \frac{(\sqrt{x}+4)^3}{\sqrt{x}}\,dx$ $\Big($Let $u = \sqrt{x}+4 = x^{1/2}+4 \xrightarrow{D}$

$du = \frac{1}{2}x^{-1/2}dx \rightarrow 2du = \frac{1}{\sqrt{x}}dx\Big)$

$= 2\int u^3 du = 2 \cdot \frac{1}{4}u^4 + C$

$= \frac{1}{2}(\sqrt{x}+4)^4 + C$

10.) (10 pts.) Consider all possible rectangles inscribed in the region below. Find the dimensions and area of the rectangle of Maximum Area.




Area

$A = xy = x\sqrt{6-x}$

$\xrightarrow{D}$ $A' = x \cdot \frac{1}{2}(6-x)^{-1/2}(-1) + (1)(6-x)^{1/2}$

$= \frac{-x}{2(6-x)^{1/2}} + \frac{(6-x)^{1/2}}{1} \cdot \frac{2(6-x)^{1/2}}{2(6-x)^{1/2}}$

$= \frac{-x+2(6-x)}{2(6-x)^{1/2}} = \frac{12-3x}{2(6-x)^{1/2}} = 0 \rightarrow 12-3x=0$

$\rightarrow x=4$

$+ \quad 0 \quad -$ $A'$

$x=4$

MAX $A = 4\sqrt{2}$