Math 16B (Winter 2021)
Kouba
Exam 3

Printing and signing your name below is a verification that no other person assisted you in the completion of this Exam.

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Show clear, organized supporting work for your answers. Correct answers without supporting work may not receive full credit. Use of unapproved shortcuts may not receive full credit. There are 8 pages. You must submit exactly 8 pages to Gradescope.

1.) Use partial fractions for each of the two indefinite integral.

a.) (11 pts.) $\int \frac{x+3}{x^2-4}\,dx = \int \frac{x+3}{(x-2)(x+2)}\,dx = \int \left[\frac{A}{x-2} + \frac{B}{x+2}\right] dx$

$$\left(\begin{array}{l} x+3 = A(x+2) + B(x-2) \\ \underline{\text{Let } x=2}: \ 5 = A(4) + B(0) \rightarrow \boxed{A = 5/4} \\ \underline{\text{Let } x=-2}: \ 1 = A(0) + B(-4) \rightarrow \boxed{B = -1/4} \end{array} \right)$$

$$= \int \left[\frac{5/4}{x-2} + \frac{-1/4}{x+2} \right] dx$$

$$= \frac{5}{4} \ln|x-2| + \frac{-1}{4} \ln|x+2| + c$$

b.) (11 pts.) $\int \frac{x-2}{x^3-x^2}\,dx = \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \right] dx$

$$\left(\begin{array}{l} x-2 = Ax(x-1) + B(x-1) + Cx^2 \\ \underline{\text{Let } x=0}: \ -2 = A(0) + B(-1) + C(0) \rightarrow \boxed{B=2} \\ \underline{\text{Let } x=1}: \ -1 = A(0) + B(0) + C(1) \rightarrow \boxed{C=-1} \\ \underline{\text{Let } x=2}: \ 0 = A(2) + (2)(1) + (-1)(4) \rightarrow \boxed{A=1} \end{array} \right)$$

$$= \int \left[\frac{1}{x} + \frac{2}{x^2} + \frac{-1}{x-1} \right] dx$$

$$= \ln|x| - \frac{2}{x} - \ln|x-1| + c$$

2.) Determine the value of each of the following three improper integrals (continued on the next page).

a.) (11 pts.) $\int_{-\infty}^{0} 2xe^{-x^2}\ dx$

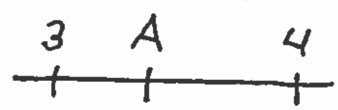
$$= \lim_{A \to -\infty} \int_A^0 2xe^{-x^2} dx$$

$$= \lim_{A \to -\infty} -e^{-x^2} \Big|_A^0$$

$$= \lim_{A \to -\infty} \left(-e^0 - -e^{-A^2}\right) = -1 + \cancel{e^{-\infty}}^{\,0} = -1$$

b.) (11 pts.) $\int_{3}^{4} \frac{1}{(x-3)^2}\ dx$

$$= \lim_{A \to 3^+} \int_A^4 (x-3)^{-2} dx$$



$$= \lim_{A \to 3^+} \frac{-1}{x-3} \Big|_A^4$$

$$= \lim_{A \to 3^+} \left(-1 - \frac{-1}{A-3}\right) = -1 + \frac{1}{0^+} = -1 + \infty$$

$$= \infty$$

c.) (11 pts.) $\int_3^\infty \frac{2}{x^2-4}\,dx$ $= \lim_{A\to\infty} \int_3^A \left[\frac{A}{x-2} + \frac{B}{x+2}\right] dx$

$$\left(\begin{array}{l} A(x+2) + B(x-2) = 2 \\ \underline{\text{Let } x=2}:\ 4A = 2 \to A = 1/2 \\ \underline{\text{Let } x=-2}:\ -4B = 2 \to B = -1/2 \end{array} \right)$$

$$= \lim_{A\to\infty} \int_3^A \left[\frac{1/2}{x-2} + \frac{-1/2}{x+2}\right] dx = \lim_{A\to\infty} \left(\tfrac{1}{2}\ln|x-2| - \tfrac{1}{2}\ln|x+2|\right)\Big|_3^A$$

$$= \lim_{A\to\infty} \left(\left(\underbrace{\tfrac{1}{2}\ln|A-2|}_{\infty} \underbrace{-}_{-} \underbrace{\tfrac{1}{2}\ln|A+2|}_{\infty}\right) - \left(\tfrac{1}{2}\overbrace{\cancel{\ln 1}}^{0} - \tfrac{1}{2}\ln 5\right)\right)$$

$$= \lim_{A\to\infty} \frac{1}{2}\ln\left|\frac{A-2}{A+2}\cdot\frac{1/A}{1/A}\right| + \frac{1}{2}\ln 5 = \lim_{A\to\infty} \ln\left|\frac{1-\overbrace{2/A}^{0}}{1+\underbrace{2/A}_{0}}\right| + \frac{1}{2}\ln 5$$

$$= \overbrace{\cancel{\ln 1}}^{0} + \frac{1}{2}\ln 5 = \frac{1}{2}\ln 5$$

3.) (11 pts.) Compute S_4, the Simpson Estimate using $n = 4$, for $\int_{-3}^{5} \sqrt{2x+6}\ dx$.

$$f(x) = \sqrt{2x+6}$$

$$h = \frac{5-(-3)}{4} = 2$$

$$S_4 = \frac{h}{3}\left[f(-3) + 4f(-1) + 2f(1) + 4f(3) + f(5)\right]$$

$$= \frac{2}{3}\left[\sqrt{0} + 4\sqrt{4} + 2\sqrt{8} + 4\sqrt{12} + \sqrt{16}\right]$$

$$= \frac{2}{3}\left[8 + 4\sqrt{2} + 8\sqrt{3} + 4\right]$$

$$= \frac{2}{3}\left[12 + 4\sqrt{2} + 8\sqrt{3}\right] \approx 21.01$$

4.) (11 pts.) Use $\displaystyle\int \frac{1}{\sqrt{u^2 \pm a^2}}\,du = \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$

to integrate $\displaystyle\int \frac{1}{\sqrt{x^2 + 10x + 29}}\,dx.$

$$\int \frac{1}{\sqrt{x^2+10x+29}}\,dx = \int \frac{1}{\sqrt{x^2+10x+25+4}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+5)^2+2^2}}\,dx \quad (\text{Let } u = x+5 \xrightarrow{D} du = dx, \text{ and } a = 2)$$

$$= \ln\left|(x+5) + \sqrt{(x+5)^2+2^2}\right| + C$$

5.) Determine the following two indefinite integrals. This problem is continued on the next page.

a.) (12 pts.) $\int x(\ln x)^2 \, dx$

(Let $u = (\ln x)^2$, $dv = x\,dx$ $\rightarrow$
$du = 2\ln x \cdot \frac{1}{x}$, $v = \frac{1}{2}x^2$)

$= \frac{1}{2}x^2(\ln x)^2 - \int x \ln x \, dx$ (Let $u = \ln x$, $dv = x\,dx$
$\rightarrow du = \frac{1}{x}dx$, $v = \frac{1}{2}x^2$)

$= \frac{1}{2}x^2(\ln x)^2 - \left[\frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx\right]$

$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{2}\cdot\frac{1}{2}x^2 + C$

b.) (12 pts.) $\boxed{\int \sin 3x \cos 5x \; dx}$

(Let $u = \sin 3X$, $dv = \cos 5X\, dX$

$\rightarrow du = 3\cos 3X\, dX$, $v = \frac{1}{5}\sin 5X$)

$= \frac{1}{5}\sin 3X \sin 5X - \frac{3}{5}\int \cos 3X \sin 5x\, dX$

(Let $u = \cos 3X$, $dv = \sin 5x\, dx$

$\rightarrow du = -3\sin 3X\, dX$, $v = \frac{-1}{5}\cos 5X$)

$= \frac{1}{5}\sin 3X \sin 5X$

$- \frac{3}{5}\left[\frac{-1}{5}\cos 3X \cos 5X - \frac{3}{5}\int \sin 3X \cos 5x\, dX\right]$

$= \frac{1}{5}\sin 3X \sin 5X + \frac{3}{25}\cos 3X \cos 5X$

$+ \frac{9}{25}\boxed{\int \sin 3X \cos 5X\, dX}$

($A = B + D - \frac{9}{25}A$, so TWIST) $\rightarrow$

$\frac{16}{25}\int \sin 3X \cos 5x\, dX = \frac{1}{5}\sin 3X \sin 5X$

$+ \frac{3}{25}\cos 3X \cos 5X + C \rightarrow$

$\int \sin 3X \cos 5x\, dX$

$= \frac{25}{16}\left[\frac{1}{5}\sin 3X \sin 5X + \frac{3}{25}\cos 3X \cos 5X + C\right]$

6.) (12 pts.) What should n be in order that Simpson's Rule estimates the exact value of the following definite integral with absolute error at most 0.0005 ? See the Error Formula on the front of this exam. $\Big($ Assume that the fourth derivative of $f(x) = \dfrac{x-1}{3-x}$ is $f^{(4)}(x) = \dfrac{48}{(3-x)^5}$. $\Big)$: $\displaystyle\int_{-2}^{1} \frac{x-1}{3-x}\,dx$

$$h = \frac{1-(-2)}{n} = \frac{3}{n} ,$$

$$\max_{-2\le x\le 1} |f^{(4)}(x)| = \max_{-2\le x\le 1} \frac{48}{|3-x|^5} = \frac{48}{|3-(1)|^5}$$

$$= \frac{48}{32} = \frac{3}{2} \; ; \text{ then}$$

$$|E_n| \le (1-(-2))\,\frac{\left(\frac{3}{n}\right)^4}{180}\left\{\frac{3}{2}\right\}$$

$$= \frac{729}{360}\cdot\frac{1}{n^4} \le 0.0005 \rightarrow$$

$$n^4 \ge \frac{729}{360(0.0005)} \rightarrow$$

$$n \ge \left[\frac{729}{360(0.0005)}\right]^{1/4} = 4050^{1/4} \approx 7.98 \text{ so}$$

choose $\boxed{n=8}$.

7.) (12 pts.) Determine if the following Improper Integral Converges or Diverges. Show clear steps/explanations: $\int_0^1 \frac{1}{x \ln x}\, dx$

($\div$ by zero at $x=0$, $x=1$)

$$= \int_0^{1/2} \frac{1}{x \ln x}\, dx + \int_{1/2}^{1} \frac{1}{x \ln x}\, dx$$

$$= B + C \;;$$

$$B = \int_0^{1/2} \frac{1}{x \ln x}\, dx = \lim_{A \to 0^+} \int_A^{1/2} \frac{1}{x \ln x}\, dx$$

$$= \lim_{A \to 0^+} \ln|\ln x| \Big|_A^{1/2}$$

$$= \lim_{A \to 0^+} \left(\ln|\ln 1/2| - \ln|\ln A| \right)$$

$$= \ln|\ln \tfrac{1}{2}| - \ln|\ln 0^+|$$

$$= \ln|\ln \tfrac{1}{2}| - \ln|-\infty|$$

$$= \ln|\ln \tfrac{1}{2}| - \ln(\infty)$$

$$= \ln|\ln \tfrac{1}{2}| - \infty = -\infty,$$

so C doesn't matter and $\int_0^1 \frac{1}{x \ln x}\, dx$ diverges.