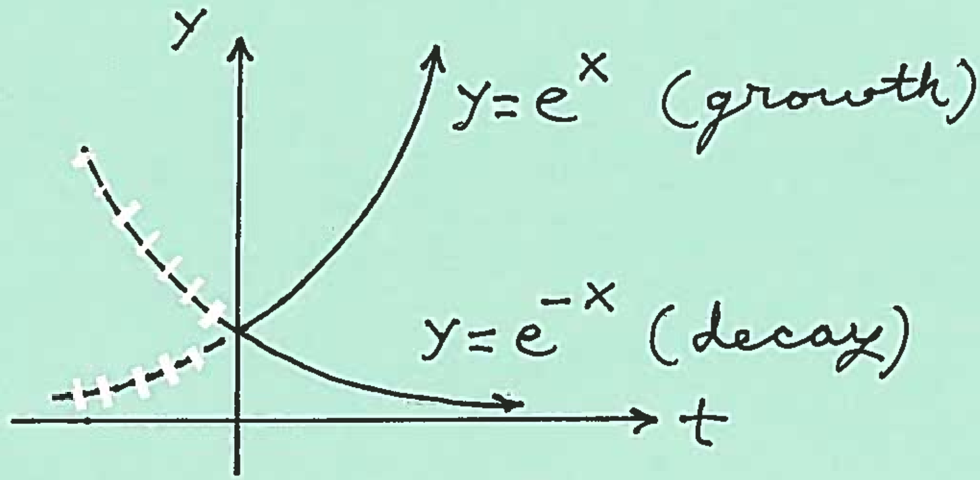
Math 16B
Section 4.6

Exponential Growth/Decay

RECALL:



Problem: assume that the rate at which quantity $A=A(t)$ changes with respect to time t is directly proportional to the quantity itself, then:

RECALL: I.) B is directly proportional to C means:

$$B = kC$$

II.) $D \ln f(x) = \frac{1}{f(x)} f'(x)$

GIVEN: $\frac{dA}{dt} = kA \rightarrow$

$$\frac{1}{A} A' = k \rightarrow$$

(derivative "backwards")

$$\ln(A) = kt + C \rightarrow$$

$$A = e^{kt+C} \rightarrow$$

$$A = e^{kt} e^{C} \rightarrow$$

$$\boxed{A = Ce^{kt}}$$

This is the Exponential Growth/Decay Equation

NOTE: If $t = 0$, then

$A = Ce^{k(0)} = Ce^{0} = C \cdot (1) = C$, so

$\boxed{\text{C is the initial amount}}$.

Example: a deer population initially has 100 deer. In 5 years there will be 165 deer. assuming exponential growth

1.) how many deer will there be in 10 years?

2.) how long will it take for there to be 500 deer?

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assume $A = Ce^{kt}$; initial amount $C = 100$, so

$$A = 100e^{kt} \quad ;$$

if $t = 5$, then $A = 165 \longrightarrow$

$165 = 100e^{k(5)} \longrightarrow \frac{165}{100} = e^{5k} \longrightarrow$

$\ln 1.65 = \ln e^{5k} \longrightarrow$

$\ln 1.65 = 5k \rightarrow k = \frac{1}{5}\ln 1.65 \rightarrow$

$$\boxed{A = 100e^{\left(\frac{1}{5}\ln 1.65\right)t}} \quad ;$$

1.) If $t=10$, then

$$A = 100\, e^{\left(\frac{1}{5}\ln 1.65\right)(10)} \approx 272 \text{ deer}$$

2.) If $A = 500$, then

$$500 = 100\, e^{\left(\frac{1}{5}\ln 1.65\right)t} \longrightarrow$$

$$5 = e^{\left(\frac{1}{5}\ln 1.65\right)t} \longrightarrow$$

$$\ln 5 = \ln e^{\left(\frac{1}{5}\ln 1.65\right)t} \longrightarrow$$

$$\ln 5 = \left(\frac{1}{5}\ln 1.65\right)t \longrightarrow$$

$$t = \frac{5\ln 5}{\ln 1.65} \approx 16.1 \text{ years}$$

Example: The # of students enrolled at UC Davis was 16,432 in Fall 1975 and had grown to 20,147 in Fall 1986. assuming exponential growth, how many students will be enrolled in Fall 2021 ?

assume $A = Ce^{kt}$ and

$t = 0$ (1975), $A = 16{,}432$;

$t = 11$ (1986), $A = 20{,}147$;

Find: A when $t = 46$ (2021) ;

Initial amount $C = 16{,}432 \longrightarrow$

$$A = 16{,}432\, e^{kt} \quad ;$$

if $t = 11$, $A = 20{,}147 \longrightarrow$

$$20{,}147 = 16{,}432\, e^{k(11)} \longrightarrow$$

$$\frac{20{,}147}{16{,}432} = e^{11k} \longrightarrow$$

$$\ln\left(\frac{20{,}147}{16{,}432}\right) = \ln e^{11k} \longrightarrow$$

$$\ln\left(\frac{20{,}147}{16{,}432}\right) = 11k \longrightarrow$$

$$k = \frac{1}{11} \ln\left(\frac{20{,}147}{16{,}432}\right) \longrightarrow$$

$$\boxed{A = 16{,}432\, e^{\left(\frac{1}{11}\ln\left(\frac{20{,}147}{16{,}432}\right)\right)t}}$$

If $t = 46$, then

$$A = 16{,}432\, e^{\left(\frac{1}{11}\ln\left(\frac{20{,}147}{16{,}432}\right)\right)(46)}$$

$\approx 38{,}536$ students

<u>Optional Question</u>: Using the previous example, what was the student enrollment in 1970?

1970 is $t = -5$ (!), so

$$A = 16{,}432\, e^{\left(\frac{1}{11}\ln\left(\frac{20{,}147}{16{,}432}\right)\right)(-5)}$$

$\approx 14{,}978$ students

FACT: Assume that the amount of quantity A decreases in such a way that its HALF LIFE is m years. This means that in m years A will lose HALF of its current value. If C is the initial amount of A then

$$t=0: \quad A=C$$

$$t=m: \quad A=\frac{1}{2}C$$

$$t=2m: \quad A=\frac{1}{2}\left(\frac{1}{2}C\right)=\frac{1}{4}C$$

$$t=3m: \quad A=\frac{1}{2}\left(\frac{1}{4}C\right)=\frac{1}{8}C$$

$$\vdots \qquad \qquad \vdots$$

Example: Assume that the sugar in your bubble gum decays exponentially and the sugar's half-life is 1.5 minutes. If you start with 80 grams of sugar

1.) how much sugar is left after 7 minutes?

2.) how long will it take for the amount of sugar to reduce to 30 grams?

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assume $A = Ce^{kt}$; initial amount $C = 80$, so

$$A = 80e^{kt};$$

if $t = 1.5$, then $A = 40$ ($\frac{1}{2}$-life) $\rightarrow$

$$40 = 80e^{k(1.5)} \rightarrow$$

$$\frac{1}{2} = e^{1.5k} \rightarrow$$

$$\ln\left(\frac{1}{2}\right) = \ln e^{1.5k} \longrightarrow$$

$$\ln(1/2) = 1.5k \longrightarrow$$

$$k = \frac{\ln(1/2)}{1.5} \longrightarrow$$

$$\boxed{A = 80e^{\frac{\ln(1/2)}{1.5}t}}$$

1.) If $t = 7$, then $A = 80e^{\frac{\ln(1/2)}{1.5}(7)}$

≈ 3.15 grams

2.) If $A = 30$, then

$$30 = 80e^{\frac{\ln(1/2)}{1.5}t} \longrightarrow$$

$$\frac{3}{8} = e^{\frac{\ln(1/2)}{1.5}t} \longrightarrow$$

$$\ln\left(\frac{3}{8}\right) = \ln e^{\frac{\ln(1/2)}{1.5}t} \longrightarrow$$

$$\ln\left(\frac{3}{8}\right) = \frac{\ln(1/2)}{1.5}t \longrightarrow$$

$$t = \frac{1.5 \ln(3/8)}{\ln(1/2)} \approx 2.12 \text{ min.}$$

Carbon Dating,

a Specific Type of $\frac{1}{2}$-Life Problem

Math 16B
Kouba
Carbon Dating

In 1960 the American scientist W. F. Libby won the Nobel prize for his discovery of carbon dating, a method for determining the age of certain fossils. Carbon dating is based on the fact that nitrogen is converted to radioactive carbon-14 by cosmic radiation in the upper atmosphere. This radioacative carbon is absorbed by plant and animal tissue through the life processes (for example, through respiration) while the plant or animal lives. However, when the plant or animal dies the absorption process stops and the amount of carbon-14 decreases (exponentially) through radioacative decay.

When the object, such as a piece of wood or bone, was part of a living organism, it accumulated small amounts of radioactive carbon-14, so that a certain proportion of the carbon in the object was carbon-14. Once the organism dies, it no longer picks up carbon-14 through interaction with its environment. By measuring the proportion of carbon-14 in the fossilized object, comparing that to the proportion in living material, and using the fact that the half-life of carbon-14 is about 5730 years, the age of the object can be estimated.

Half-life of C-14 is 5730 years.

Example : If an oak tree dies today, then how much C-14 remains in its fossilized remains after 500 years ?

assume $A = Ce^{kt}$, where C is the initial amount of carbon-14 (when tree dies) ; and if $t = 5730$, then $A = \frac{1}{2}C$ ($\frac{1}{2}$-life) $\rightarrow$

$$\frac{1}{2}C = Ce^{k(5730)} \rightarrow$$

$$\frac{1}{2} = e^{5730k} \rightarrow$$

$$\ln\left(\frac{1}{2}\right) = \ln e^{5730k} \rightarrow$$

$$\ln\left(\frac{1}{2}\right) = 5730k \rightarrow$$

$$k = \frac{\ln(1/2)}{5730} \rightarrow$$

$$\boxed{A = Ce^{\frac{\ln(1/2)}{5730}t}} \;;$$

if $t = 500 \rightarrow$

$$A = Ce^{\frac{\ln(1/2)}{5730}(500)}$$

$$\approx C\,(0.9413)$$

$= 94.13\%$ of C, the original amount of C-14.

<u>Example</u>: A fossilized shark tooth contains 7.35% of its original amount of carbon-14. Estimate the age of the shark tooth.

assume $A = Ce^{kt}$, where C is the original amount of carbon-14; and if $t = 5730$, then $A = \frac{1}{2}C$ ($\frac{1}{2}$-life)

$$\rightarrow \quad \tfrac{1}{2}C = Ce^{k(5730)}$$

$$\rightarrow \quad \tfrac{1}{2} = e^{5730k}$$

$$\rightarrow \quad \ln\left(\tfrac{1}{2}\right) = \ln e^{5730k}$$

$$\to \quad \ln\left(\frac{1}{2}\right) = 5730\,k$$

$$\to \quad k = \frac{\ln(1/2)}{5730}$$

$$\to \quad \boxed{A = Ce^{\frac{\ln(1/2)}{5730}t}} \quad ;$$

if $A = 7.35\%$ of $C = 0.0735C \to$

$$0.0735C = Ce^{\frac{\ln(1/2)}{5730}t} \to$$

$$0.0735 = e^{\frac{\ln(1/2)}{5730}t} \to$$

$$\ln 0.0735 = \ln e^{\frac{\ln(1/2)}{5730}t} \to$$

$$\ln 0.0735 = \frac{\ln(1/2)}{5730}t \to$$

$$t = \frac{5730(\ln 0.0735)}{\ln(1/2)} \approx 21{,}580 \text{ yrs.}$$