Math 16B
Section 5.3

Integrating Exponential Functions

RECALL: $De^x = e^x$

RULE: $\int e^x dx = e^x + c$

Example: $\int e^{5x} dx$

(Let $u = 5x \xrightarrow{D} du = 5\,dx \rightarrow \frac{1}{5}du = dx$)

$= \frac{1}{5}\int e^u du = \frac{1}{5}e^u + c = \frac{1}{5}e^{5x} + c$

RULE: $\int e^{kx} dx = \frac{1}{k}e^{kx} + c$

Example: Integrate.

1.) $\int (e^x + e^{3x} + e^{-2x} + e^{\frac{3}{2}x})\,dx$

$= e^x + \frac{1}{3}e^{3x} + \frac{-1}{2}e^{-2x} + \frac{2}{3}e^{\frac{3}{2}x} + c$

2.) $\int \frac{e^{5x}+3}{e^{3x}}\,dx = \int \left[\frac{e^{5x}}{e^{3x}} + \frac{3}{e^{3x}}\right] dx$

$$= \int (e^{5x-3x} + 3e^{-3x})\,dx = \int (e^{2x} + 3e^{-3x})\,dx$$

$$= \frac{1}{2}e^{2x} + 3\cdot\frac{-1}{3}e^{-3x} + c$$

3.) $\int (1-e^x)^2\,dx = \int (1 - 2e^x + e^{2x})\,dx$

$$= x - 2e^x + \frac{1}{2}e^{2x} + c$$

4.) $\int e^x(2+e^x)^5\,dx$

(Let $u = 2 + e^x \xrightarrow{D} du = e^x\,dx$)

$$= \int u^5\,du = \frac{1}{6}u^6 + c$$

5.) $\int \frac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx$

(Let $u = 1 + e^{2x} \xrightarrow{D} du = 2e^{2x}\,dx \rightarrow$
$\frac{1}{2}du = e^{2x}\,dx$)

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du$$

$$= \frac{1}{2}\cdot 2u^{1/2} + c = \sqrt{1+e^{2x}} + c$$

6.) $\int (e^x - e^{-x})(e^{2x} + e^{-3x})\,dx$

$= \int (e^x e^{2x} + e^x e^{-3x} - e^{-x} e^{2x} - e^{-x} e^{-3x})\,dx$

$= \int (e^{3x} + e^{-2x} - e^x - e^{-4x})\,dx$

$= \frac{1}{3} e^{3x} + \frac{-1}{2} e^{-2x} - e^x - \frac{-1}{4} e^{-4x} + c$

7.) $\int x \cos(x^2) e^{\sin(x^2)}\,dx$

(Let $u = \sin(x^2) \xrightarrow{D} du = 2x \cdot \cos(x^2)\,dx$

$\rightarrow \frac{1}{2} du = x \cos(x^2)\,dx$)

$= \frac{1}{2} \int e^u\,du = \frac{1}{2} e^u + c = \frac{1}{2} e^{\sin(x^2)} + c$

8.) $\int \frac{e^{3x}}{(3+e^{3x})^3}\,dx$

(Let $u = 3 + e^{3x} \xrightarrow{D} du = 3e^{3x}\,dx \rightarrow$

$\frac{1}{3} du = e^{3x}\,dx$)

$= \frac{1}{3} \int \frac{1}{u^3}\,du = \frac{1}{3} \int u^{-3}\,du = \frac{1}{3} \cdot \frac{-1}{2} u^{-2} + c$

$= \frac{-1}{6} (3 + e^{3x})^{-2} + c$

Logarithmic Antiderivatives

RECALL: $D \ln x = \frac{1}{x}$

What is the derivative of $\ln|x|$?

we know $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, then

$$D \ln|x| = \begin{cases} D \ln x, & \text{if } x > 0 \\ D \ln(-x), & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} 1/x, & \text{if } x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x}, & \text{if } x < 0 \end{cases} \text{, i.e.,}$$

$$D \ln|x| = \frac{1}{x} \longrightarrow$$

RULE: $\int \frac{1}{x}\,dx = \ln|x| + C$

Example: Integrate.

1.) $\int \frac{x+1}{x^2}\,dx = \int \left[\frac{x}{x^2} + \frac{1}{x^2}\right] dx$

$= \int \left[\frac{1}{x} + x^{-2}\right] dx = \ln|x| - x^{-1} + C$

2.) $\int \frac{x}{x^2+4}\,dx$

(Let $u = x^2+4 \xrightarrow{D} du = 2x\,dx \rightarrow$
$\frac{1}{2}du = x\,dx$)

$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c$

$= \frac{1}{2}\ln|x^2+4| + c$

3.) $\int \frac{e^x}{e^x+4}\,dx$

(Let $u = e^x+4 \xrightarrow{D} du = e^x\,dx$)

$= \int \frac{1}{u}\,du = \ln|u| + c = \ln|e^x+4| + c$

4.) $\int \frac{1}{x\ln x}\,dx$

(Let $u = \ln x \xrightarrow{D} du = \frac{1}{x}\,dx$)

$= \int \frac{1}{u}\,du = \ln|u| + c$

$= \ln|\ln x| + c$

5.) $\int \frac{7}{5x+4}\,dx$

(Let $u = 5x+4 \xrightarrow{D} du = 5\,dx \rightarrow \frac{1}{5}du = dx$)

$= 7\cdot\frac{1}{5}\int \frac{1}{u}\,du = \frac{7}{5}\ln|u| + c$

$= \frac{7}{5}\ln|5x+4| + c$

6.) $\int \frac{x^2}{x+1}\,dx$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \\ & -(x^2+x) \\ \hline & -x \\ & -(-x-1) \\ \hline & 1 \end{array}$$

$= \int \left[x - 1 + \frac{1}{x+1}\right] dx$

$= \frac{1}{2}x^2 - x + \ln|x+1| + c$

* 7.) $\int \frac{1}{x+\sqrt{x}}\,dx = \int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$

(Let $u = \sqrt{x}+1 \xrightarrow{D} du = \frac{1}{2}x^{-1/2}dx \rightarrow$

$2\,du = \frac{1}{\sqrt{x}}\,dx$)

$= 2\int \frac{1}{u}\,du = 2\ln|u| + c$

$= 2\ln|\sqrt{x}+1| + c$

8.) $\int \frac{x^2+1}{x^3+3x-5}\,dx$

(Let $u = x^3+3x-5 \xrightarrow{D} du = (3x^2+3)\,dx$

$\longrightarrow du = 3(x^2+1)\,dx \longrightarrow \frac{1}{3}du = (x^2+1)\,dx$)

$= \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C$

$= \frac{1}{3}\ln|x^3+3x-5| + C$

9.) $\int \frac{\sec^2 x}{3+\tan x}\,dx$

(Let $u = 3+\tan x \xrightarrow{D} du = \sec^2 x\,dx$)

$= \int \frac{1}{u}\,du = \ln|u| + C$

$= \ln|3+\tan x| + C$

10.) $\displaystyle\int \frac{x^3+x^2+2x+1}{x^2+1}\,dx$

$$
\begin{array}{r|l}
 & \quad x+1 \\
\hline
x^2+1 & x^3+x^2+2x+1 \\
 & -(x^3 \quad + x) \\
\hline
 & \quad x^2+x+1 \\
 & -(x^2+1) \\
\hline
 & \quad x
\end{array}
$$

$$= \int \left[x+1 + \frac{x}{x^2+1} \right] dx$$

$$= \frac{1}{2}x^2 + x + \frac{1}{2}\ln\left|x^2+1\right| + c$$

RULE: $\int \tan x \, dx = \ln|\sec x| + c$

PROOF: $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

(Let $u = \cos x \xrightarrow{D} du = -\sin x \, dx \longrightarrow$
$-du = \sin x \, dx$)

$= -\int \frac{1}{u} \, du = -\ln|u| + c$

$= -\ln|\cos x| + c = \ln|\cos x|^{-1} + c$

$= \ln\left|\frac{1}{\cos x}\right| + c = \ln|\sec x| + c$

RULE: $\int \sec x \, dx = \ln|\sec x + \tan x| + c$

PROOF: $\int \sec x \, dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$

(Let $u = \sec x + \tan x \xrightarrow{D} du = (\sec x \tan x + \sec^2 x) \, dx$)

$= \int \frac{1}{u} \, du = \ln|u| + c$

$= \ln|\sec x + \tan x| + c$