Math 16B
Section 5.6

Estimating the Value of a Definite Integral

Suppose that the integral $\int_a^b f(x)\,dx$ is too difficult (or impossible) to compute, or that you are simply required to estimate its exact value. The following two methods offer two different ways to determine good estimates.

1.) MIDPOINT RULE

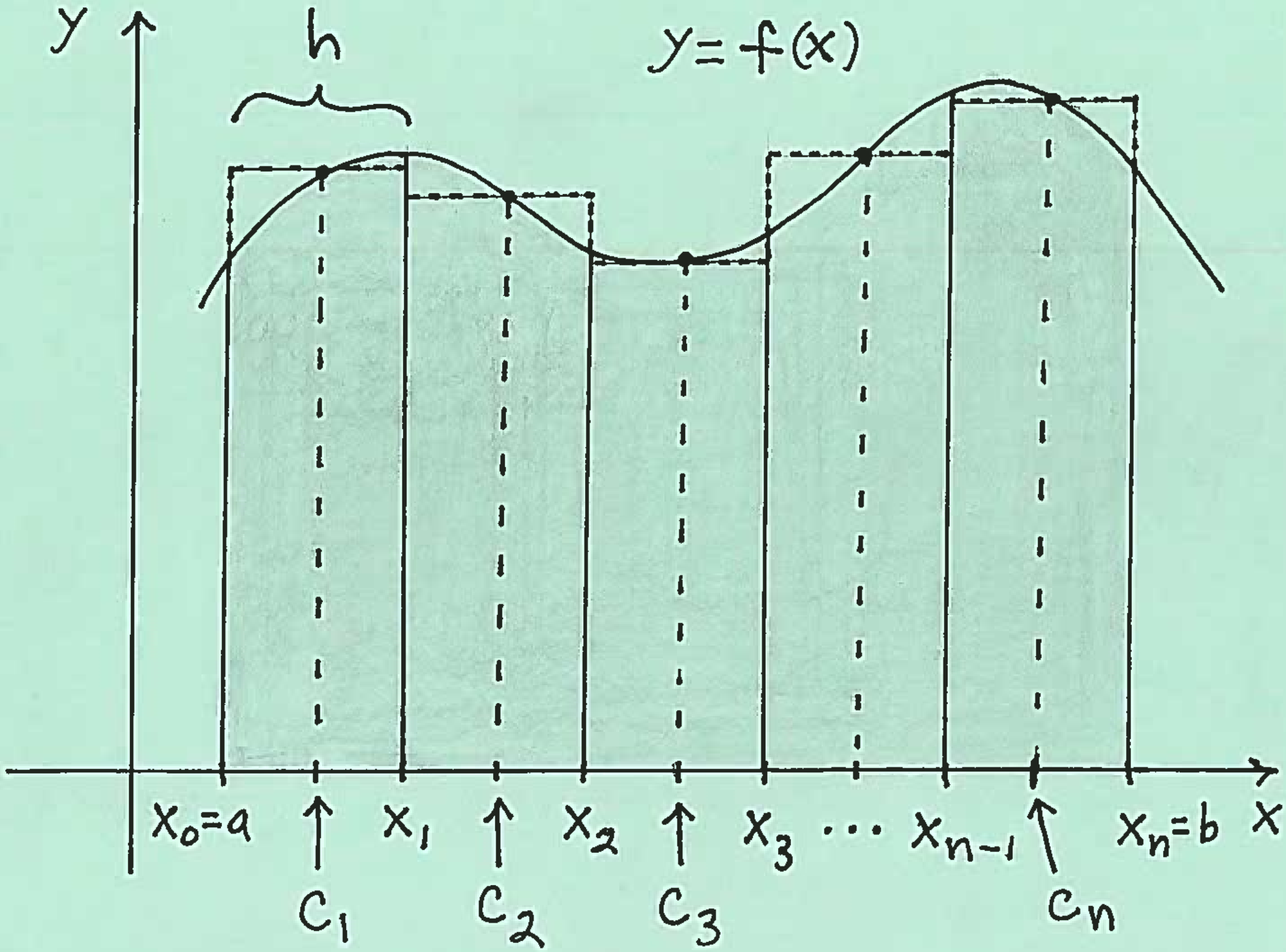
a.) Divide the interval $[a, b]$ into n equal parts, each of length $h = \dfrac{b-a}{n}$.

b.) Let $a = x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n = b$ be the partition of the interval and let the sampling points $c_1, c_2, c_3, ..., c_n$ be the MIDPOINTS of these subintervals.

c.) The Midpoint Estimate for $\int_a^b f(x)\,dx$ is

$$M_n = h\,[f(c_1) + f(c_2) + f(c_3) + ... + f(c_n)]\ .$$

(SEE diagram on the next page.)



Shaded AREA ≈ the sum of the areas of the n RECTANGLES, i.e.,

$$\int_a^b f(x)\,dx \approx h f(c_1) + h f(c_2) + h f(c_3) + \cdots + h f(c_n)$$

$$= h\,[f(c_1) + f(c_2) + f(c_3) + \cdots + f(c_n)$$

$$= M_n \quad , \text{ where } h = \frac{b-a}{n}$$

2.) TRAPEZOIDAL RULE

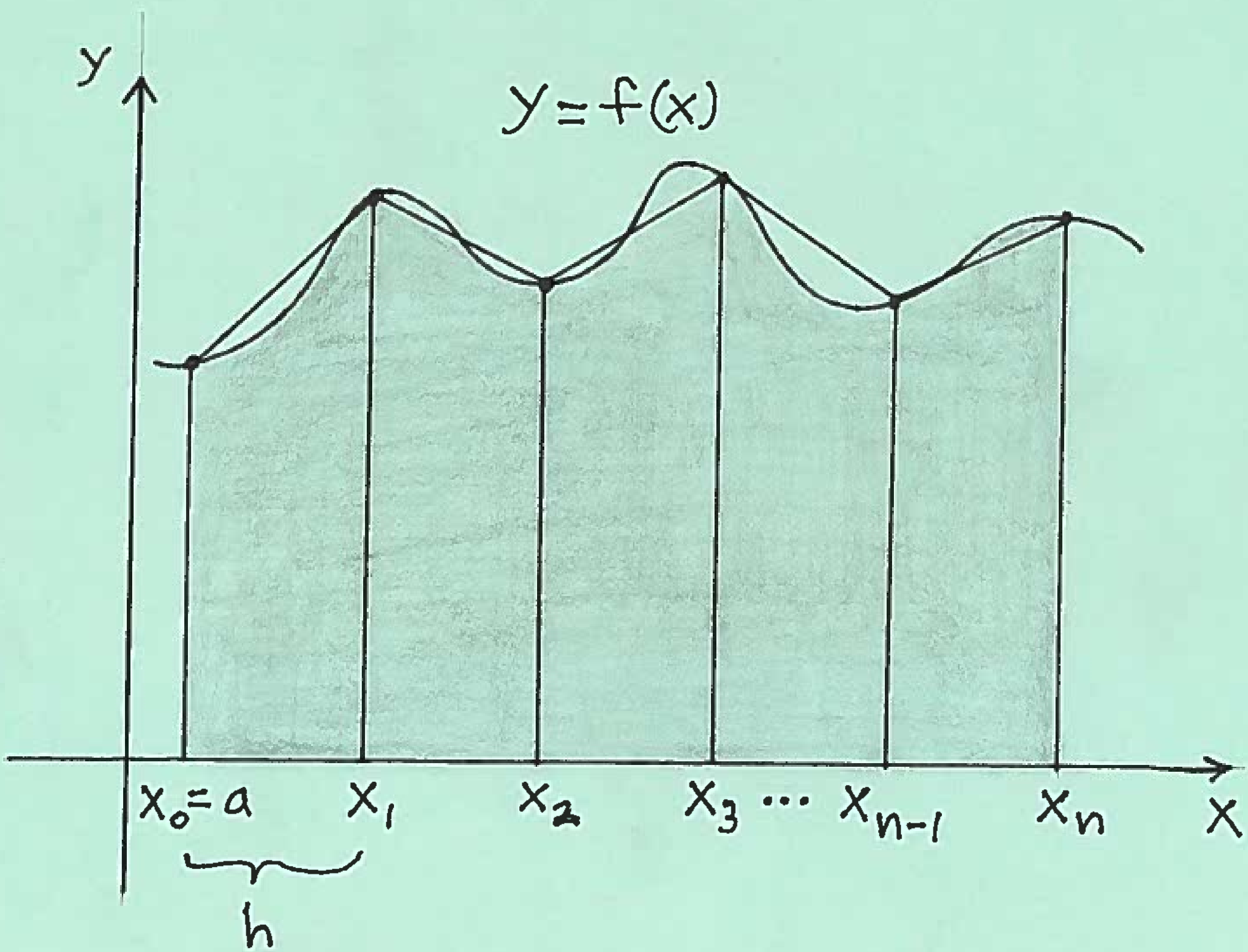
a.) Divide the interval $[a, b]$ into n equal parts, each of length $h = \dfrac{b-a}{n}$.

b.) Let $a = x_0, x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b$ be the partition of the interval.

c.) The Trapezoidal Estimate for $\int_a^b f(x)\,dx$ is

$$T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right].$$

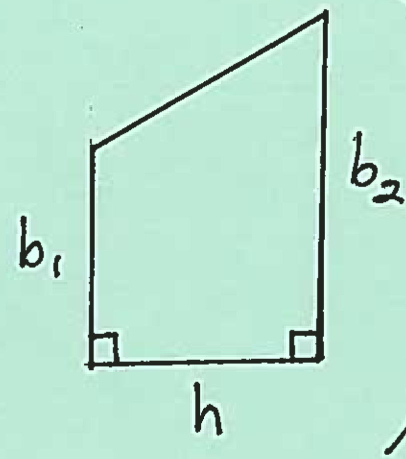
(SEE diagram below.)



Shaded AREA $\approx$ the sum of the areas of the n TRAPEZOIDS, i.e.,

(RECALL: AREA of TRAPEZOID is

$$A = \frac{1}{2}(b_1 + b_2)h$$



)

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(f(x_0)+f(x_1))h + \frac{1}{2}(f(x_1)+f(x_2))h + \frac{1}{2}(f(x_2)+f(x_3))h + \cdots + \frac{1}{2}(f(x_{n-2})+f(x_{n-1}))h + \frac{1}{2}(f(x_{n-1})+f(x_n))h$$

$$= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

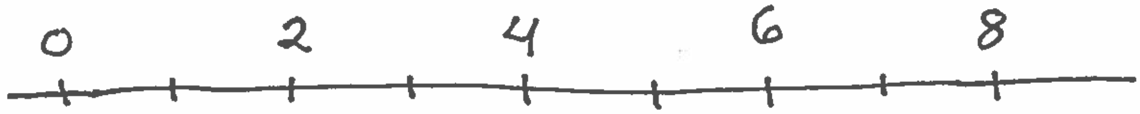
$$= T_n \quad , \text{ where } h = \frac{b-a}{n}$$

Example: Use the Midpoint Rule with the given n to estimate the value of

$$\int_0^8 \sqrt{x+4}\; dx .$$

1.) $n=4$

2.) $n=8$

1.)


midpoints: 1 3 5 7

$h = \frac{8-0}{4} = 2$, $f(x) = \sqrt{x+4}$, and

$$M_4 = h\,[f(1) + f(3) + f(5) + f(7)]$$

$$= 2\,[\sqrt{5} + \sqrt{7} + \sqrt{9} + \sqrt{11}\,] \approx 22.3969$$

2.)


mdpts: $\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$ $\frac{9}{2}$ $\frac{11}{2}$ $\frac{13}{2}$ $\frac{15}{2}$

$h = \frac{8-0}{8} = 1$, $f(x) = \sqrt{x+4}$, and

$$M_8 = h\left[f\left(\tfrac{1}{2}\right) + f\left(\tfrac{3}{2}\right) + f\left(\tfrac{5}{2}\right) + \cdots + f\left(\tfrac{15}{2}\right)\right]$$

$$= 1\left[\sqrt{\tfrac{1}{2}+4} + \sqrt{\tfrac{3}{2}+4} + \sqrt{\tfrac{5}{2}+4} + \cdots + \sqrt{\tfrac{15}{2}+4}\right]$$

$$= \sqrt{\tfrac{9}{2}} + \sqrt{\tfrac{11}{2}} + \sqrt{\tfrac{13}{2}} + \sqrt{\tfrac{15}{2}} + \sqrt{\tfrac{17}{2}} + \sqrt{\tfrac{19}{2}} + \sqrt{\tfrac{21}{2}} + \sqrt{\tfrac{23}{2}}$$

$$\approx 23.3839$$

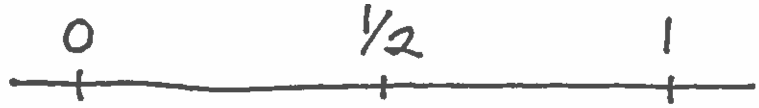
<u>CALCULATOR</u>: $\int_0^8 \sqrt{x+4}\,dx \approx 22.3795$

<u>Example</u>: Use the Trapezoidal Rule with the given n to estimate the value of

$$\int_0^1 \frac{1}{x^2+1}\,dx.$$

1.) $n = 2$

2.) $n = 5$

1.) 

$$h = \frac{1-0}{2} = \frac{1}{2}, \quad f(x) = \frac{1}{x^2+1}, \text{ and}$$

$$T_2 = \frac{h}{2}\left[f(0) + 2f\left(\tfrac{1}{2}\right) + f(1)\right]$$

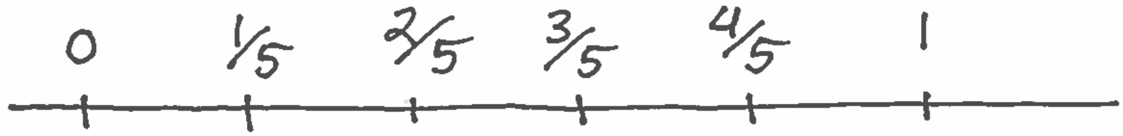
$$= \frac{\frac{1}{2}}{2}\left[1 + 2\cdot\frac{1}{\frac{1}{4}+1} + \frac{1}{2}\right]$$

$$= \frac{1}{4}\left[1 + 2\cdot {}^{1}/_{\frac{5}{4}} + \frac{1}{2}\right]$$

$$= \frac{1}{4}\left[1 + \frac{8}{5} + \frac{1}{2}\right]$$

$$= \frac{1}{4}\left[\frac{10}{10} + \frac{16}{10} + \frac{5}{10}\right]$$

$$= \frac{1}{4}\left[\frac{31}{10}\right] = \frac{31}{40} = 0.775$$

2.) 

$$h = \frac{1-0}{5} = \frac{1}{5}, \quad f(x) = \frac{1}{x^2+1}, \text{ and}$$

$$T_5 = \frac{h}{2}\Big[f(0) + 2f\left({}^{1}/_{5}\right) + 2f\left({}^{3}/_{5}\right)$$

$$+ 2f\left({}^{3}/_{5}\right) + 2f\left({}^{4}/_{5}\right) + f(1)\Big]$$

$$= \frac{\frac{1}{5}}{2}\left[1 + 2\cdot\frac{1}{\frac{1}{25}+1} + 2\cdot\frac{1}{\frac{4}{25}+1} + 2\cdot\frac{1}{\frac{9}{25}+1} + 2\cdot\frac{1}{\frac{16}{25}+1} + \frac{1}{2}\right]$$

$$= \frac{1}{10}\left[1 + 2\cdot\frac{1}{\frac{26}{25}} + 2\cdot\frac{1}{\frac{29}{25}} + 2\cdot\frac{1}{\frac{34}{25}} + 2\cdot\frac{1}{\frac{41}{25}} + \frac{1}{2}\right]$$

$$= \frac{1}{10}\left[1 + \frac{50}{26} + \frac{50}{29} + \frac{50}{34} + \frac{50}{41} + \frac{1}{2}\right]$$

$$\approx 0.7837$$

<u>CALCULATOR</u>: $\int_0^1 \frac{1}{x^2+1}\,dx \approx 0.7854$