Math 16B
Section 6.2

Integration by Parts More Than Once

Example: Integrate.

1.) $\int x^2 \sin x \, dx$

(Let $u = x^2$, $dv = \sin x \, dx$
$\rightarrow du = 2x \, dx$, $v = -\cos x$)

$= -x^2 \cos x - -2 \int x \cos x \, dx$

(Let $u = x$, $dv = \cos x \, dx$
$\rightarrow du = dx$, $v = \sin x$)

$= -x^2 \cos x + 2 \left[x \sin x - \int \sin x \, dx \right]$

$= -x^2 \cos x + 2x \sin x - 2(-\cos x) + C$

2.) $\int (\ln x)^2 dx$

$$\left(\text{Let } u = (\ln x)^2, \quad dv = dx \right.$$

$$\left. \longrightarrow du = 2 \ln x \cdot \frac{1}{x}, \quad v = x \right)$$

$$= x(\ln x)^2 - 2\int x \cdot \frac{1}{x} \ln x \, dx$$

$$= x(\ln x)^2 - 2\int \ln x \, dx$$

$$\left(\text{Let } u = \ln x, \quad dv = dx \right.$$

$$\left. \longrightarrow du = \frac{1}{x} dx, \quad v = x \right)$$

$$= x(\ln x)^2 - 2\left[x \ln x - \int x \cdot \frac{1}{x} dx\right]$$

$$= x(\ln x)^2 - 2x \ln x - 2\int 1 \, dx$$

$$= x(\ln x)^2 - 2x \ln x - 2x + C$$

3.) $\int x^3 e^x \, dx$

$\left(\text{Let } u = x^3, \; dv = e^x dx \right.$

$\left. \longrightarrow du = 3x^2 dx, \; v = e^x \right)$

$= x^3 e^x - 3 \int x^2 e^x \, dx$

$\left(\text{Let } u = x^2, \; dv = e^x dx \right.$

$\left. \longrightarrow du = 2x \, dx, \; v = e^x \right)$

$= x^3 e^x - 3 \left[x^2 e^x - 2 \int x e^x \, dx \right]$

$= x^3 e^x - 3x^2 e^x - 6 \int x e^x \, dx$

$\left(\text{Let } u = x, \; dv = e^x dx \right.$

$\left. \longrightarrow du = dx, \; v = e^x \right)$

$= x^3 e^x - 3x^2 e^x - 6 \left[x e^x - \int e^x \, dx \right]$

$= x^3 e^x - 3x^2 e^x - 6x e^x + 6 e^x + c$

Integration by Parts with a TWIST

Example: Integrate.

1.) $\int e^x \cos x \, dx$

$$\left(\text{Let } u = e^x,\ dv = \cos x \, dx \right.$$
$$\left. \rightarrow du = e^x \, dx,\ v = \sin x \right)$$

$$= e^x \sin x - \int e^x \sin x \, dx$$

$$\left(\text{Let } u = e^x,\ dv = \sin x \, dx \right.$$
$$\left. \rightarrow du = e^x \, dx,\ v = -\cos x \right)$$

$$= e^x \sin x - \left[-e^x \cos x - {}^{-}\int e^x \cos x \, dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx \text{, i.e.,}$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

(TWIST: $A = B + D - A \rightarrow$

$2A = B + D \rightarrow$

$A = \frac{1}{2} B + \frac{1}{2} D$)

TWIST

$$\longrightarrow 2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C$$

$$\longrightarrow \int e^x \cos x\,dx = \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + C$$

2.) $\int e^x \sin 3x\,dx$

(Let $u = \sin 3x$, $dv = e^x\,dx$

$\longrightarrow du = 3\cos 3x\,dx$, $v = e^x$)

$$= e^x \sin 3x - 3\int e^x \cos 3x\,dx$$

(Let $u = \cos 3x$, $dv = e^x\,dx$

$\longrightarrow du = {}^-3\sin 3x$, $v = e^x$)

$$= e^x \sin 3x - 3\left[e^x \cos 3x - {}^-3\int e^x \sin 3x\,dx\right]$$

$$= e^x \sin 3x - 3e^x \cos 3x - 9\int e^x \sin 3x\,dx$$

(TWIST : $A = B - D - 9A \longrightarrow$

$10A = B - D \longrightarrow$

$A = \frac{1}{10}B - \frac{1}{10}D$)

TWIST

$$\longrightarrow 10\int e^x \sin 3x\,dx = e^x \sin 3x - 3e^x \cos 3x + C$$

$\longrightarrow \int e^x \sin 3x \, dx = \frac{1}{10} e^x \sin 3x - \frac{3}{10} e^x \cos 3x + C$

3.) $\int \cos x \cdot \sin 4x \, dx$

(Let $u = \cos x$, $dv = \sin 4x \, dx$

$\longrightarrow du = -\sin x \, dx$, $v = \frac{-1}{4} \cos 4x$)

$= \frac{-1}{4} \cos x \cos 4x - \frac{1}{4} \int \sin x \cos 4x \, dx$

(Let $u = \sin x$, $dv = \cos 4x \, dx$

$\longrightarrow du = \cos x \, dx$, $v = \frac{1}{4} \sin 4x$)

$= \frac{-1}{4} \cos x \cos 4x - \frac{1}{4} \left[\frac{1}{4} \sin x \sin 4x - \frac{1}{4} \int \cos x \sin 4x \, dx \right]$

$= \frac{-1}{4} \cos x \cos 4x - \frac{1}{16} \sin x \sin 4x$

$+ \frac{1}{16} \int \cos x \sin 4x \, dx$

TWIST

$\longrightarrow \left(1 - \frac{1}{16}\right) \int \cos x \sin 4x \, dx$

$= \frac{-1}{4} \cos x \cos 4x - \frac{1}{16} \sin x \sin 4x + C$

$$\longrightarrow \frac{15}{16}\int \cos x \sin 4x \, dx$$

$$= \frac{-1}{4}\cos x \cos 4x - \frac{1}{16}\sin x \sin 4x + C$$

$$\longrightarrow \int \cos x \sin 4x \, dx$$

$$= \frac{16}{15}\left[\frac{-1}{4}\cos x \cos 4x - \frac{1}{16}\sin x \sin 4x + C\right]$$

$$\longrightarrow \int \cos x \sin 4x \, dx$$

$$= -\frac{4}{15}\cos x \cos 4x - \frac{1}{15}\sin x \sin 4x + C$$