Math 16B
Section 6.3

Integration Using Partial Fractions

Example: $\frac{11}{12} = \frac{11}{(3)(4)}$

$$= \frac{8}{(3)(4)} + \frac{3}{(3)(4)}$$

$$= \frac{2}{3} + \frac{1}{4}$$

Let's call these partial fractions.
Will this idea work for rational functions? Try it on the following example.

$$\int \frac{x+1}{x^2-x}\,dx = \int \frac{x+1}{x(x-1)}\,dx$$

$$= \int \left[\frac{A}{x} + \frac{B}{x-1} \right] dx \;;$$

(Let's solve for unknown constants A and B :

$$\frac{A}{x}+\frac{B}{x-1}=\frac{A}{x}\cdot\frac{x-1}{x-1}+\frac{x}{x}\cdot\frac{B}{x-1}$$

$$=\frac{A(x-1)+Bx}{x(x-1)} \leftarrow$$

$$=\frac{x+1}{x(x-1)} \leftarrow$$

so $\boxed{x+1=A(x-1)+Bx}$

<u>Let $x=1$</u>: $2=A(0)+B(1) \longrightarrow \boxed{B=2}$

<u>Let $x=0$</u>: $1=A(-1)+B(0) \longrightarrow \boxed{A=-1}$

$$=\int\left[\frac{-1}{x}+\frac{2}{x-1}\right]dx$$

$$=-\ln|x|+2\ln|x-1|+C$$

<u>Example</u>: Use Partial Fractions to integrate.

1.) $\int\frac{1}{x^2-4}\,dx=\int\frac{1}{(x-2)(x+2)}\,dx$

$$= \int \left[\frac{A}{x-2} + \frac{B}{x+2} \right] dx$$

$$\left(\boxed{1 = A(x+2) + B(x-2)} \right.$$

<u>Let $x=2$</u>: $1 = A(4) + B(0) \rightarrow \boxed{A = \frac{1}{4}}$

<u>Let $x=-2$</u>: $1 = A(0) + B(-4) \rightarrow \boxed{B = \frac{-1}{4}}$)

$$= \int \left[\frac{1/4}{x-2} + \frac{-1/4}{x+2} \right] dx$$

$$= \frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| + C$$

2.) $\displaystyle\int \frac{x+2}{x^2+x-6}\, dx = \int \frac{x+2}{(x-2)(x+3)}\, dx$

$$= \int \left[\frac{A}{x-2} + \frac{B}{x+3} \right] dx$$

$$\left(\boxed{x+2 = A(x+3) + B(x-2)} \right.$$

<u>Let $x=2$</u>: $4 = A(5) + B(0) \rightarrow \boxed{A = 4/5}$

<u>Let $x=-3$</u>: $-1 = A(0) + B(-5) \rightarrow \boxed{B = 1/5}$)

$$= \int \left[\frac{4/5}{x-2} + \frac{1/5}{x+3} \right] dx$$

$$= \frac{4}{5} \ln|x-2| + \frac{1}{5} \ln|x+3| + C$$

3.) $$\int \frac{x^2 - x + 2}{x^3 - x}\, dx = \int \frac{x^2 - x + 2}{x(x^2-1)}\, dx$$

$$= \int \frac{x^2 - x + 2}{x(x-1)(x+1)}\, dx = \int \left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \right] dx$$

$\Big($ $x^2 - x + 2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

Let $x=1$: $2 = A(0) + B(2) + C(0) \rightarrow \boxed{B=1}$

Let $x=-1$: $4 = A(0) + B(0) + C(2) \rightarrow \boxed{C=2}$

Let $x=0$: $2 = A(-1) + B(0) + C(0) \rightarrow \boxed{A=-2}$ $\Big)$

$$= \int \left[\frac{-2}{x} + \frac{1}{x-1} + \frac{2}{x+1} \right] dx$$

$$= -2\ln|x| + \ln|x-1| + 2\ln|x+1| + C$$

Partial Fractions and Repeated Factors

4.) $\int \frac{x+6}{x^3-3x^2}\,dx = \int \frac{x+6}{x^2(x-3)}\,dx$

(Notice that factor "x" occurs twice.)

$= \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} \right] dx$

$\Big(x+6 = Ax(x-3) + B(x-3) + Cx^2$

<u>Let $x=0$</u>: $6 = A(0) + B(-3) + C(0)$

$\rightarrow \boxed{B = -2}$

<u>Let $x=3$</u>: $9 = A(0) + B(0) + C(9)$

$\rightarrow \boxed{C = 1}$

<u>Let $x=2$</u>: $8 = A(-2) + (-2)(-1) + (1)(4)$

$\rightarrow 8 = -2A + 2 + 4 \rightarrow 2 = -2A \rightarrow \boxed{A = -1} \Big)$

$= \int \left[\frac{-1}{x} + \frac{-2}{x^2} + \frac{1}{x-3} \right] dx$

$$= -\ln|x| + 2x^{-1} + \ln|x-3| + C$$

5.) $\displaystyle\int \frac{x^2+3}{x(x-1)^2}\,dx = \int\left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\right]dx$

($x^2+3 = A(x-1)^2 + Bx(x-1) + Cx$

<u>Let $x=0$</u>: $3 = A(1) + B(0) + C(0)$

$\longrightarrow \boxed{A=3}$

<u>Let $x=1$</u>: $4 = A(0) + B(0) + C(1)$

$\longrightarrow \boxed{C=4}$

<u>Let $x=2$</u>: $7 = (3)(1) + B(2) + (4)(2)$

$\longrightarrow 7 = 3 + 2B + 8 \rightarrow -4 = 2B \rightarrow \boxed{B=-2}$)

$$= \int\left[\frac{3}{x} + \frac{-2}{x-1} + \frac{4}{(x-1)^2}\right]dx$$

$$= 3\ln|x| - 2\ln|x-1| - 4(x-1)^{-1} + C$$

Polynomial Division Before Using Partial Fractions

(FACT: If the degree of the numerator is equal to or greater than the degree of the denominator, then polynomial division should be the first action.)

6.) $\int \frac{x^3+1}{x^2-9}\,dx$

$$\begin{array}{r l} & x \\ x^2-9 \,\big) & x^3+1 \\ & -(x^3-9x) \\ \hline & 9x+1 \end{array}$$

$$= \int \left[x + \frac{9x+1}{x^2-9}\right] dx$$

$$= \frac{1}{2}x^2 + \int \frac{9x+1}{(x-3)(x+3)}\,dx$$

$$= \frac{1}{2}x^2 + \int \left[\frac{A}{x-3} + \frac{B}{x+3}\right] dx$$

$$\left(9x+1 = A(x+3) + B(x-3) \right.$$

$\underline{\text{Let } x=3}$: $28 = A(6) + B(0)$

$$\rightarrow A = \frac{28}{6} = \frac{14}{3} \rightarrow \boxed{A = \frac{14}{3}}$$

$\underline{\text{Let } x=-3}$: $-26 = A(0) + B(-6)$

$$\left. \rightarrow B = \frac{26}{6} = \frac{13}{3} \rightarrow \boxed{B = \frac{13}{3}} \right)$$

$$= \frac{1}{2}x^2 + \int \left[\frac{14/3}{x-3} + \frac{13/3}{x+3} \right] dx$$

$$= \frac{1}{2}x^2 + \frac{14}{3}\ln|x-3| + \frac{13}{3}\ln|x+3| + C$$