Math 16B
Section 6.6

Improper Integrals

Definition: An integral $\int_a^b f(x)\,dx$ is *improper* if

i.) $a = \pm\infty$

ii.) $b = \pm\infty$

iii.) f is not continuous at some x-value in the interval $[a,b]$ (e.g., division by zero)

Example: All of these are improper integrals.

1.) $\int_0^{\infty} x e^{-x^2}\,dx$; $b = \infty$

2.) $\int_{-\infty}^{0} e^{2x}\,dx$; $a = -\infty$

3.) $\int_{4}^{5} \frac{3}{\sqrt{x-4}}\,dx$; f is not cont. at $x=4$.

4.) $\int_{-\infty}^{\infty} \frac{1}{x^2-1}\,dx$; $a=-\infty$, $b=\infty$, f is not cont. at $x=1$, $x=-1$.

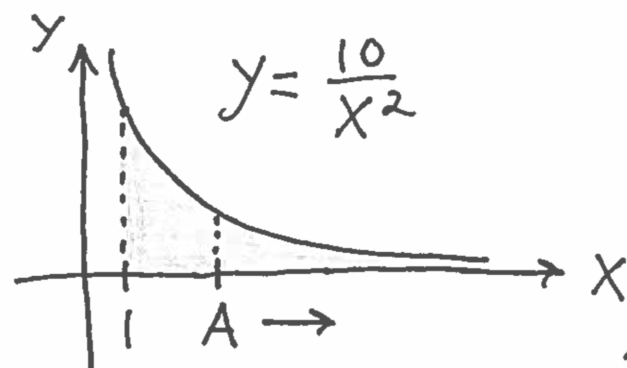
<u>Example</u>: Determine the value of each Improper Integral.

<u>TERMINOLOGY</u>:

I.) If an answer is FINITE, we say the improper integral CONVERGES.

II.) If an answer is NOT FINITE, we say the improper integral DIVERGES.

1.) $\int_{1}^{\infty} \frac{10}{x^2}\,dx$



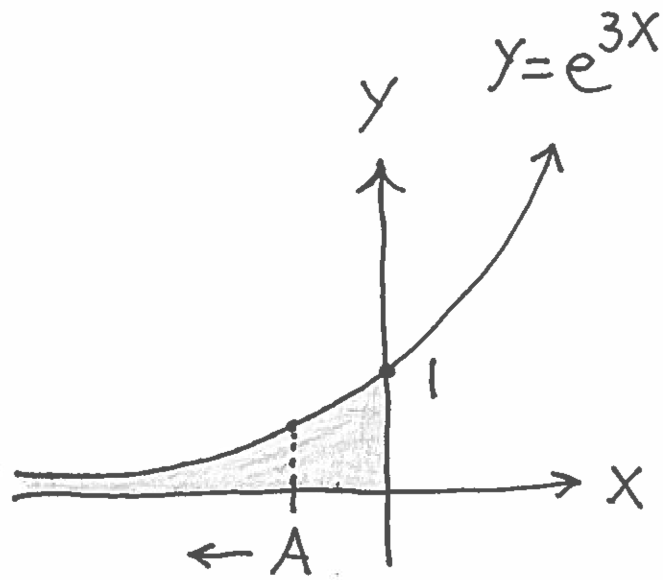
$$= \lim_{A\to\infty} \int_1^A \frac{10}{x^2}\,dx$$

$$= \lim_{A\to\infty} \left.-\frac{10}{x}\right|_1^A$$

$$= \lim_{A\to\infty} \left(-\frac{10}{A} - \frac{-10}{1}\right)$$

$$= 0 + 10 = 10$$

2.) $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx$



$$= \lim_{A\to-\infty} \int_A^0 e^{3x}\,dx$$

$$= \lim_{A\to-\infty} \left.\frac{1}{3}e^{3x}\right|_A^0$$

$$= \lim_{A\to-\infty} \left(\frac{1}{3}e^{0} - \frac{1}{3}e^{3A}\right)$$

$$= \lim_{A\to-\infty} \left(\frac{1}{3} - \frac{1}{3}e^{3A}\right)$$

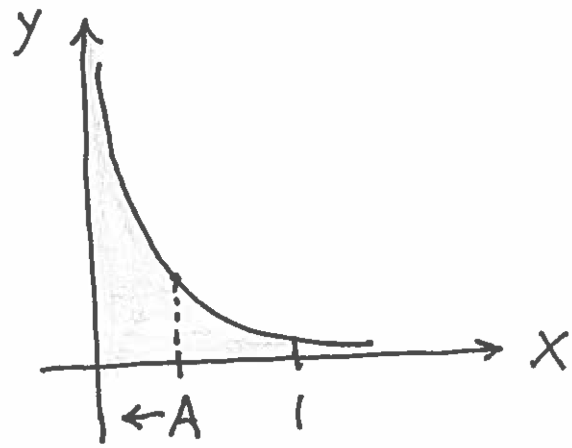
$$= \frac{1}{3} - \frac{1}{3} e^{-\infty}$$

$$= \frac{1}{3} - \frac{1}{3} \cdot \frac{1}{e^{\infty}}$$

$$= \frac{1}{3} - \frac{1}{3} \cdot \frac{1}{\infty}$$

$$= \frac{1}{3} - \frac{1}{3}(0) = \frac{1}{3}$$

3.) $\displaystyle\int_0^1 \frac{1}{x}\,dx$



$$= \lim_{A \to 0^+} \int_A^1 \frac{1}{x}\,dx$$

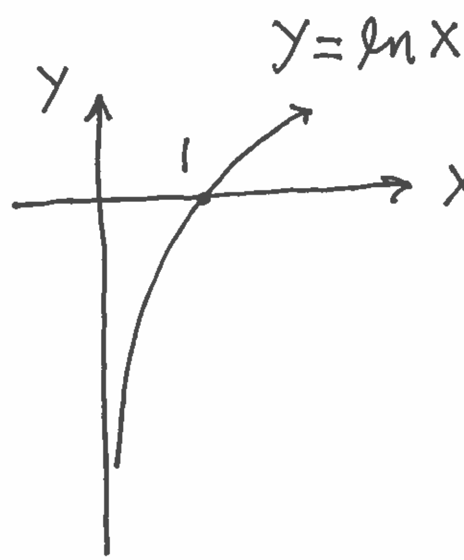
$$= \lim_{A \to 0^+} \ln|x| \Big|_A^1$$

$$= \lim_{A \to 0^+} \left(\overset{0}{\cancel{\ln 1}} - \ln A\right)$$

$$= -\ln(0^+)$$

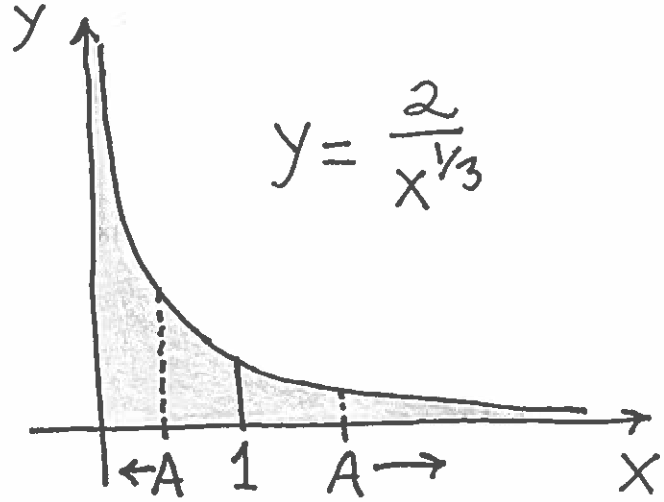
$$= -(-\infty)$$

$$= \infty$$



4.) $\int_0^\infty \frac{2}{x^{1/3}}\,dx$

(÷ by 0 AND $b=\infty$; do them separately)



$$= \int_0^1 \frac{2}{x^{1/3}}\,dx + \int_1^\infty \frac{2}{x^{1/3}}\,dx$$

$$= \quad B + C \quad ;$$

$$B = \int_0^1 \frac{2}{x^{1/3}}\,dx = \int_0^1 2\cdot x^{-1/3}\,dx$$

$$= \lim_{A\to 0^+} \int_A^1 2\cdot x^{-1/3}\,dx$$

$$= \lim_{A\to 0^+} \cancel{2}\cdot\frac{3}{\cancel{2}}\,x^{2/3}\Big|_A^1$$

$$= \lim_{A\to 0^+} \left(3(1)^{2/3} - 3(A)^{2/3}\right)$$

$$= 3(1) - 3(0)^{2/3} = 3 \quad ;$$

$$C = \int_1^\infty \frac{2}{x^{1/3}}\,dx = \int_1^\infty 2\cdot x^{-1/3}\,dx$$

$$= \lim_{A\to\infty} \int_1^A 2\cdot x^{-1/3}\,dx$$

$$= \lim_{A\to\infty} \cancel{2}\cdot \frac{3}{\cancel{2}} x^{2/3} \Big|_1^A$$

$$= \lim_{A\to\infty} \left(3A^{2/3} - 3(1)^{2/3}\right)$$

$$= 3(\infty)^{2/3} - 3$$

$$= \infty - 3 = \infty \text{ ; thus,}$$

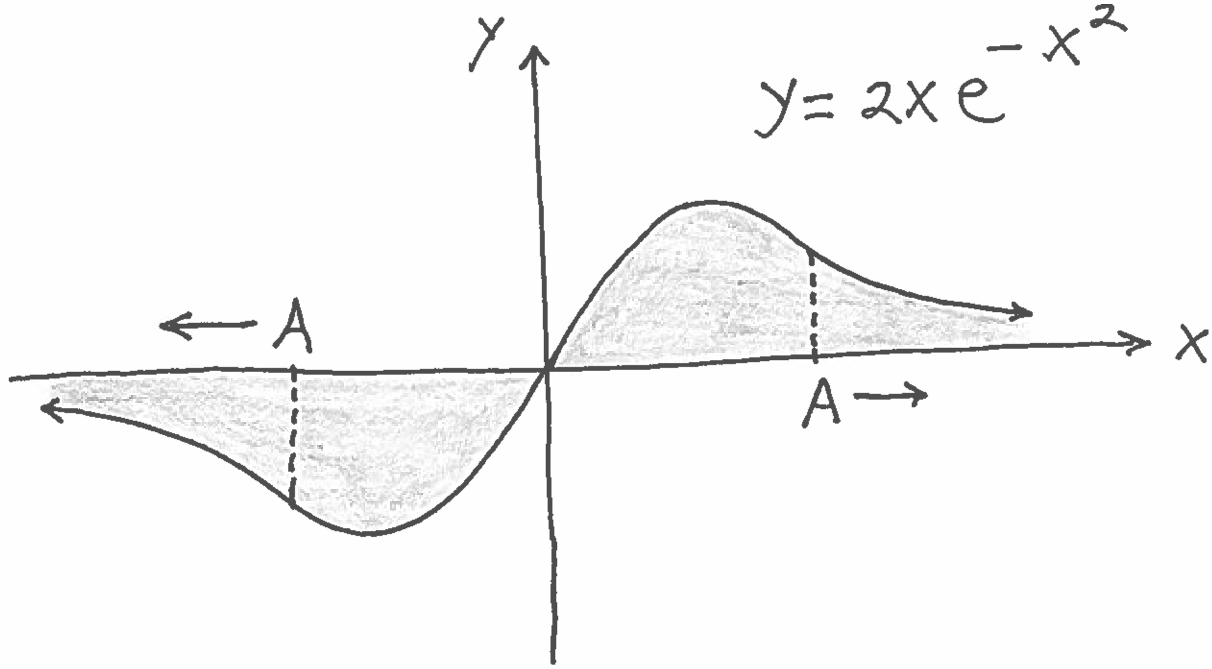
$$\int_0^\infty \frac{2}{x^{1/3}}\,dx = B + C$$

$$= 3 + \infty$$

$$= \infty$$

5.) $\int_{-\infty}^{\infty} 2x e^{-x^2}\,dx$

($a = -\infty$ and $b = \infty$; do them separately)



$$= \int_{-\infty}^{0} 2xe^{-x^2}\,dx + \int_{0}^{\infty} 2xe^{-x^2}\,dx$$

$$= \quad B + C \;;$$

$$B = \int_{-\infty}^{0} 2xe^{-x^2}\,dx$$

$$= \lim_{A \to -\infty} \int_{A}^{0} 2xe^{-x^2}\,dx$$

$$= \lim_{A \to -\infty} -e^{-x^2}\Big|_{A}^{0}$$

$$= \lim_{A \to -\infty} \left(-e^{0} - -e^{-A^2}\right)$$

$$= -1 + e^{-(-\infty)^2}$$

$$= -1 + e^{-(\infty)}$$

$$= -1 + \frac{1}{e^{\infty}}$$

$$= -1 + \frac{1}{\infty}$$

$$= -1 + 0 = -1 \ ;$$

$$C = \int_0^{\infty} 2x e^{-x^2} dx$$

$$= \lim_{A \to \infty} \int_0^{A} 2x e^{-x^2} dx$$

$$= \lim_{A \to \infty} -e^{-x^2} \Big|_0^{A}$$

$$= \lim_{A \to \infty} \left(-e^{-A^2} - -e^{0}\right)$$

$$= -e^{-(\infty)^2} + 1$$

$$= -e^{-\infty} + 1$$

$$= \frac{-1}{e^{\infty}} + 1$$

$$= \frac{-1}{\infty} + 1$$

$$= 0 + 1 = 1$$; *thus*,

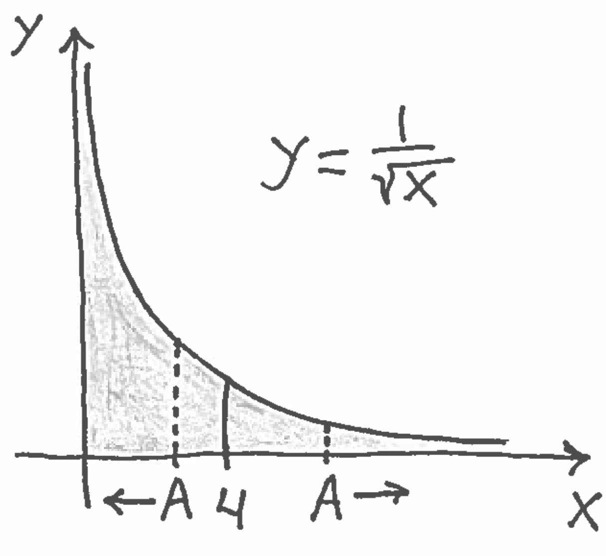
$$\int_{-\infty}^{\infty} 2x e^{-x^2} dx = B + C$$

$$= -1 + 1$$

$$= 0$$

6.) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}} dx$

(÷ by zero and $b = \infty$; do them separately)



$$= \int_0^4 \frac{1}{\sqrt{x}} dx + \int_4^\infty \frac{1}{\sqrt{x}} dx$$

$$= B + C \; ;$$

$$C = \int_4^\infty \frac{1}{\sqrt{x}} dx = \lim_{A \to \infty} \int_4^A x^{-1/2} dx$$

$$= \lim_{A\to\infty} 2x^{1/2} \Big|_4^A$$

$$= \lim_{A\to\infty} \left(2A^{1/2} - 2(4)^{1/2}\right)$$

$$= 2(\infty)^{1/2} - 2(2)$$

$$= \infty - 4$$

$= \infty$; there is no need to determine B since

$$\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = B + C$$

$$= B + \infty \quad \text{DIVERGES}$$

since $B + \infty$ CANNOT be FINITE.

(*) 7.) $\int_4^\infty \frac{1}{x^2-1}\,dx$ $\quad$ ($b=\infty$ only)

$$= \lim_{A\to\infty} \int_4^A \frac{1}{(x-1)(x+1)}\,dx$$

$$= \lim_{A\to\infty} \int_4^A \left[\frac{B}{x-1} + \frac{C}{x+1}\right] dx$$

($1 = B(x+1) + C(x-1)$

Let $x=1$: $1 = B(2) + C(0) \rightarrow \boxed{B = \frac{1}{2}}$

Let $x=-1$: $1 = B(0) + C(-2) \rightarrow \boxed{C = \frac{-1}{2}}$)

$$= \lim_{A\to\infty} \int_4^A \left[\frac{1/2}{x-1} + \frac{-1/2}{x+1}\right] dx$$

$$= \lim_{A\to\infty} \left[\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\right]\Big|_4^A$$

$$= \lim_{A\to\infty} \Big[\big(\tfrac{1}{2}\ln|A-1| - \tfrac{1}{2}\ln|A+1|\big) - \big(\tfrac{1}{2}\ln 3 - \tfrac{1}{2}\ln 5\big)\Big]$$

"$\infty - \infty$"

$$= \lim_{A\to\infty} \frac{1}{2}\ln\left|\frac{A-1}{A+1}\right| - \frac{1}{2}\ln\left(\frac{3}{5}\right)$$

"$\frac{\infty}{\infty}$"

$$= \lim_{A\to\infty} \frac{1}{2} \ln\left|\frac{A-1}{A+1}\cdot\frac{1/A}{1/A}\right| - \frac{1}{2}\ln\left(\frac{3}{5}\right)$$

$$= \lim_{A\to\infty} \frac{1}{2} \ln\left|\frac{1-1/A}{1+1/A}\right| - \frac{1}{2}\ln\left(3/5\right)$$

$$= \frac{1}{2} \ln\left|\frac{1-0}{1+0}\right| - \frac{1}{2}\ln\left(3/5\right)$$

$$= \frac{1}{2} \cancelto{0}{\ln 1} - \frac{1}{2}\ln\left(3/5\right)$$

$$= -\frac{1}{2}\ln\left(3/5\right)$$