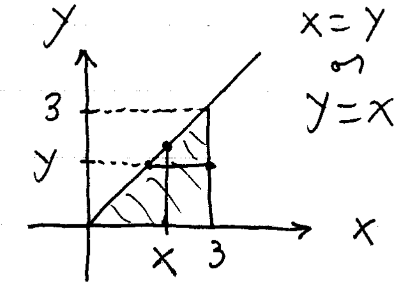
Section 7.8

33.)



$$\int_0^3 \int_y^3 e^{x^2}\,dx\,dy \quad (\text{SWITCH})$$

$$= \int_0^3 \int_0^x e^{x^2}\,dy\,dx$$

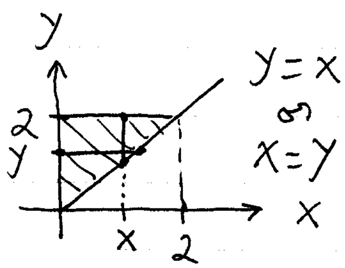
$$= \int_0^3 \left(e^{x^2}\cdot y \Big|_{y=0}^{y=x}\right) dx = \int_0^3 x e^{x^2}\,dx$$

(Let $u = x^2 \rightarrow du = 2x\,dx \rightarrow \frac{1}{2}du = x\,dx$)

$$= \int_0^3 \frac{1}{2} e^u\,du = \frac{1}{2} e^u \Big|_{x=0}^{x=3} = \frac{1}{2} e^{x^2}\Big|_0^3$$

$$= \frac{1}{2}(e^9 - e^0) = \frac{1}{2}(e^9 - 1)$$

34.)



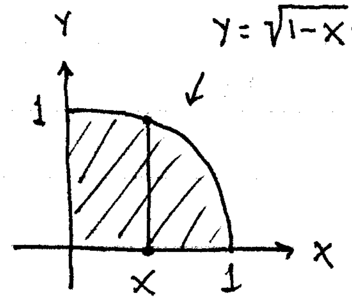
$$\int_0^2 \int_x^2 e^{-y^2}\,dy\,dx \quad (\text{SWITCH})$$

$$= \int_0^2 \int_0^y e^{-y^2}\,dx\,dy$$

$$= \int_0^2 \left(e^{-y^2}\cdot x \Big|_{x=0}^{x=y}\right) dy = \int_0^2 y e^{-y^2}\,dy$$

$$= \frac{-1}{2} e^{-y^2}\Big|_0^2 = \frac{-1}{2} e^{-4} - \frac{-1}{2}\cancel{e^0}^{1} = \frac{1}{2} - \frac{1}{2} e^{-4}$$

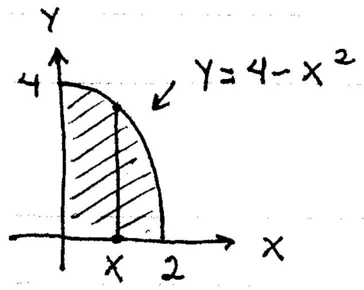
Section 7.9

5.) 

$$\int_0^1 \int_0^{\sqrt{1-x^2}} Y \, dy \, dx$$

$$= \int_0^1 \left(\frac{1}{2} Y^2 \Big|_{Y=0}^{Y=\sqrt{1-x^2}} \right) dx$$

$$= \int_0^1 \frac{1}{2}(1-x^2)\, dx = \frac{1}{2}\left(x - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{2}\left(1-\frac{1}{3}\right) = \frac{1}{3}$$

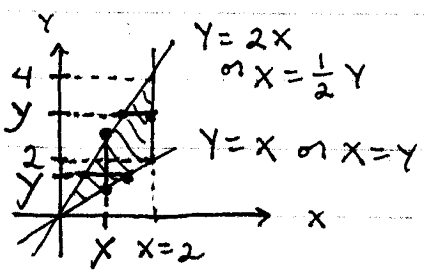
6.) 

$$\int_0^2 \int_0^{4-x^2} xy^2 \, dy \, dx$$

$$= \int_0^2 \left(x \cdot \frac{1}{3} Y^3 \Big|_{Y=0}^{Y=4-x^2} \right) dx$$

$$= \int_0^2 x \cdot \frac{1}{3}(4-x^2)^3 \, dx$$

$$= \frac{1}{3} \cdot \frac{1}{4}\left(\frac{-1}{2}\right)(4-x^2)^4 \Big|_0^2 = \frac{-1}{24}(0)^4 - \frac{-1}{24}(4)^4 = \frac{32}{3}$$

11.) 

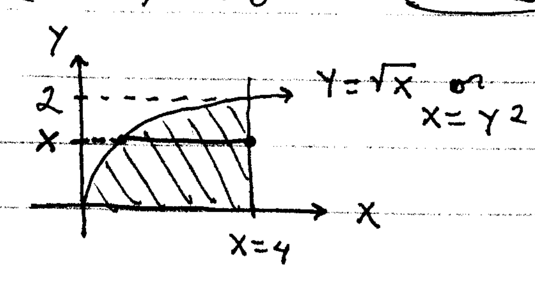
I.) $$\int_0^2 \int_{\frac{1}{2}Y}^{Y} \frac{Y}{X^2+Y^2} \, dx \, dy + \int_2^4 \int_{\frac{1}{2}Y}^{2} \frac{Y}{X^2+Y^2} \, dx \, dy$$

OR II.) $$\int_0^2 \int_X^{2X} \frac{Y}{X^2+Y^2} \, dy \, dx = \int_0^2 \left(\frac{1}{2} \ln(X^2+Y^2) \Big|_{Y=X}^{Y=2X} \right) dx$$

$$= \int_0^2 \frac{1}{2}\ln(5x^2) - \frac{1}{2}\ln(2x^2) \quad dx$$

$$= \frac{1}{2}\int_0^2 [\ln 5 + \cancel{2 \ln x} - \ln 2 - \cancel{2 \ln x}]\, dx = \frac{1}{2}\int_0^2 \left(\ln \frac{5}{2}\right) dx$$

$$= \frac{1}{2}\left(\ln 5/2\right) x \Big|_0^2 = \ln 5/2$$

12.) 

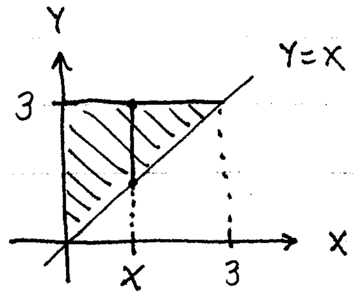
I.) $$\int_0^2 \int_{Y^2}^4 \frac{Y}{1+X^2} \, dx \, dy$$

OR

11.) $\int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx = \int_0^4 \frac{1}{1+x^2} \cdot \left(\frac{1}{2} y^2 \Big|_{y=0}^{y=\sqrt{x}} \right) dx$

$= \int_0^4 \frac{1}{2} \cdot \frac{x}{1+x^2}\,dx = \frac{1}{2} \cdot \frac{1}{2} \ln(1+x^2) \Big|_0^4 = \boxed{\frac{1}{4} \ln 17}$

17.)



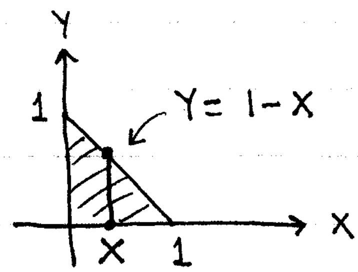
$Vol = \int_0^3 \int_x^3 (8 - x - y)\,dy\,dx$

$= \int_0^3 \left(8y - xy - \frac{1}{2}y^2\right)\Big|_{y=x}^{y=3} dx$

$= \int_0^3 \left[\left(24 - 3x - \frac{9}{2}\right) - \left(8x - x^2 - \frac{1}{2}x^2\right)\right] dx = \int_0^3 \left(\frac{3}{2}x^2 - 11x + \frac{39}{2}\right) dx$

$= \left(\frac{1}{2}x^3 - \frac{11}{2}x^2 + \frac{39}{2}x\right)\Big|_0^3 = \frac{27}{2} - \frac{99}{2} + \frac{117}{2} = \frac{45}{2}$

20.)



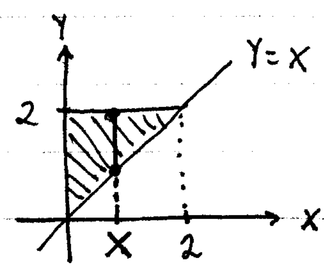
$Vol = \int_0^1 \int_0^{1-x} (1 - x - y)\,dy\,dx$

$= \int_0^1 \left(y - xy - \frac{1}{2}y^2\right)\Big|_{y=0}^{y=1-x} dx$

$= \int_0^1 \left((1-x) - x(1-x) - \frac{1}{2}(1-x)^2\right) dx = \int_0^1 \frac{1}{2}(1-x)^2\,dx$

$= \frac{-1}{6}(1-x)\Big|_0^1 = \frac{-1}{6}(0) - \frac{-1}{6}(1) = \frac{1}{6}$

22.)



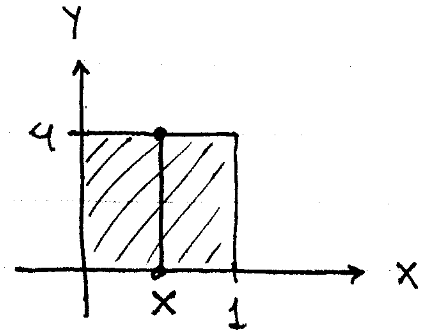
$Vol = \int_0^2 \int_x^2 (4 - y^2)\,dy\,dx$

$= \int_0^2 \left(4y - \frac{1}{3}y^3\right)\Big|_{y=x}^{y=2} dx$

$= \int_0^2 \left(\frac{16}{3} - 4x + \frac{1}{3}x^3\right) dx = \left(\frac{16}{3}x - 2x^2 + \frac{1}{12}x^4\right)\Big|_0^2$

$= \frac{32}{3} - 8 + \frac{4}{3} = 4$

27.)

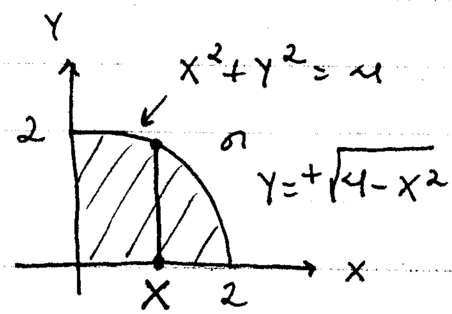


$$\text{Vol} = \int_0^1 \int_0^4 xy \, dy \, dx$$

$$= \int_0^1 \left(x \cdot \tfrac{1}{2} y^2 \Big|_{y=0}^{y=4} \right) dx$$

$$= \int_0^1 8x \, dx = 4x^2 \Big|_0^1 = 4$$

30.)



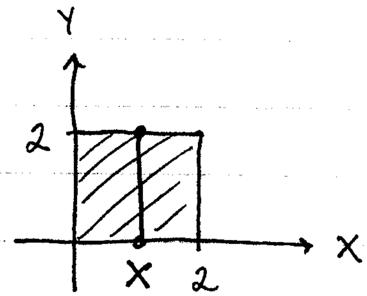
$$\text{Vol} = \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y) \, dy \, dx$$

$$= \int_0^2 \left(xy + \tfrac{1}{2} y^2 \right) \Big|_{y=0}^{y=\sqrt{4-x^2}} dx$$

$$= \int_0^2 \left(x\sqrt{4-x^2} + \tfrac{1}{2}(4-x^2) \right) dx = \left[\tfrac{-1}{3}(4-x^2)^{3/2} + \tfrac{1}{2}\left(4x - \tfrac{1}{3}x^3\right) \right] \Big|_0^2$$

$$= \tfrac{1}{2}\left(\tfrac{16}{3}\right) - \tfrac{-1}{3}(4)^{3/2} = \frac{16}{3}$$

35.)



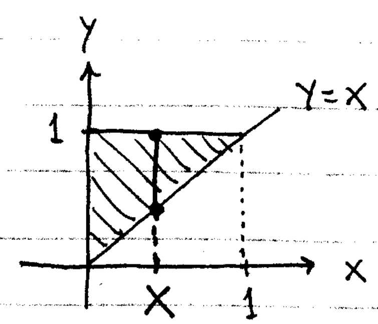
area $R = 4$ so

$$\text{Ave} = \frac{1}{\text{area } R} \int_0^2 \int_0^2 (x^2 + y^2) \, dy \, dx$$

$$= \frac{1}{4} \int_0^2 \left(x^2 y + \tfrac{1}{3} y^3 \right) \Big|_{y=0}^{y=2} dx = \frac{1}{4} \int_0^2 \left(2x^2 + \tfrac{8}{3} \right) dx$$

$$= \frac{1}{4} \left(\tfrac{2}{3}x^3 + \tfrac{8}{3}x \right) \Big|_0^2 = \frac{1}{4} \cdot \frac{32}{3} = \frac{8}{3}$$

36.)



area $R = \frac{1}{2}$ so

$$\text{Ave} = \frac{1}{\text{area } R} \int_0^1 \int_x^1 e^{x+y} \, dy \, dx$$

$$= \frac{1}{\frac{1}{2}} \int_0^1 \left(e^{X+Y} \Big|_{Y=X}^{Y=1} \right) dx$$

$$= 2 \int_0^1 \left(e^{X+1} - e^{2X} \right) dx$$

$$= 2 \left(e^{X+1} - \frac{1}{2} e^{2X} \right) \Big|_0^1$$

$$= 2 \left(e^2 - \frac{1}{2} e^2 \right) - 2 \left(e - \frac{1}{2} \right)$$

$$= \boxed{e^2 - 2e + 1}$$

Math 16C
Kouba
Worksheet 7

1.) Evaluate the following double integrals. Realize that in some cases you must switch the order of integration before you compute the antiderivatives.

a.) $\int_0^{\pi/2} \int_0^{\pi/2} \sin x \cos y \; dx \; dy$

b.) $\int_0^{2\pi} \int_0^{\pi} \cos(x/4 + y/3) \; dy \; dx$

c.) $\int_0^{1} \int_0^{\sqrt{x}} y \cdot \sin(\pi x) \; dy \; dx$

d.) $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} \; dy \; dx$

e.) $\int_0^{2\sqrt{\pi}} \int_{y/2}^{\sqrt{\pi}} \sin(x^2) \; dx \; dy$

2.) Compute the volume of the solid which lies *between* the two surfaces (draw a rough sketch) $z = x^2 + y^2 + 10$ and $x + 2y + 3z = 6$ and *above* the region R in the xy-plane bounded by the graphs of $y = 2x$ and $y = x^2$.

3.) Compute the volume of the solid which is bounded by the planes $x = 0, \; y = 0, \; z = 0,$ and $3x + 3y + 5z = 15$.

Worksheet 7

① a.) $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin x \cos y \, dx\, dy$

$= \int_0^{\frac{\pi}{2}} \left(-\cos x \cdot \cos y \Big|_{x=0}^{x=\frac{\pi}{2}}\right) dy = \int_0^{\frac{\pi}{2}} \cos y \, dy$

$= \sin y \Big|_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$

b.) $\int_0^{2\pi}\int_0^{\pi} \cos\left(\frac{x}{4} + \frac{y}{3}\right) dy\, dx$

$= \int_0^{2\pi} 3\sin\left(\frac{x}{4}+\frac{y}{3}\right)\Big|_{y=0}^{y=\pi} dx$

$= \int_0^{2\pi} \left[3\sin\left(\frac{x}{4}+\frac{\pi}{3}\right) - 3\sin\left(\frac{x}{4}\right)\right] dx$

$= -12\cos\left(\frac{x}{4}+\frac{\pi}{3}\right) + 12\cos\left(\frac{x}{4}\right) \Big|_0^{2\pi}$

$= \left(-12\cos\left(\frac{5}{6}\pi\right) + 12\cos\left(\frac{\pi}{2}\right)\right.$
$\quad - \left(-12\cos\left(\frac{\pi}{3}\right) + 12\cos(0)\right)$

$= -12\left(\frac{-\sqrt{3}}{2}\right) + 12(0) + 12\left(\frac{1}{2}\right) - 12(1) = \boxed{6\sqrt{3} - 6}$

c.) $\int_0^1\int_0^{\sqrt{x}} y\sin(\pi x)\, dy\, dx$

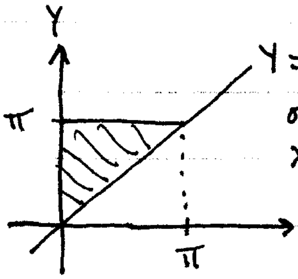
$= \int_0^1 \frac{y^2}{2}\sin(\pi x)\Big|_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^1 x\sin(\pi x)\, dx$

(Let $u = x$, $dv = \sin(\pi x)\, dx$
$du = dx$, $v = \frac{-1}{\pi}\cos(\pi x)$)

$= \frac{1}{2}\left[\frac{-x}{\pi}\cos(\pi x)\Big|_0^1 - \frac{-1}{\pi}\int_0^1 \cos(\pi x)\, dx\right]$

$= \frac{-1}{2\pi}\cos\pi + \frac{1}{\pi}\cdot\frac{1}{\pi}\sin(\pi x)\Big|_0^1$

$$= \frac{-1}{2\pi}(-1) + \frac{1}{\pi^2}\left(\overset{0}{\cancel{\sin\pi}} - \overset{0}{\cancel{\sin 0}}\right) = \frac{1}{2\pi}$$

d.) 

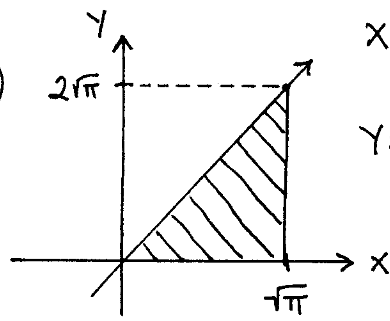
$Y = X$ or $X = Y$

$$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$$

(SWITCH ORDER OF INTEGRATION!!)

$$= \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi}\left(\frac{\sin y}{y}\cdot x\Big|_{x=0}^{x=y}\right) dy$$

$$= \int_0^{\pi} \sin y\, dy = -\cos y\Big|_0^{\pi} = -(-1)-(-1) = 2$$

e.) 

$X = \frac{Y}{2}$ or $Y = 2X$

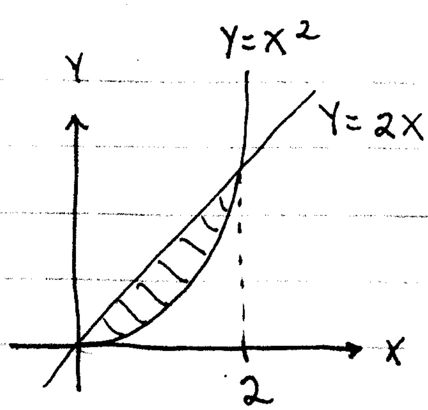
$$\int_0^{2\sqrt{\pi}}\int_{y/2}^{\sqrt{\pi}} \sin(x^2)\, dx\, dy$$

(SWITCH ORDER OF INTEGRATION!!)

$$= \int_0^{\sqrt{\pi}}\int_0^{2x} \sin(x^2)\, dy\, dx = \int_0^{\sqrt{\pi}} y\cdot\sin(x^2)\Big|_{y=0}^{y=2x} dx$$

$$= \int_0^{\sqrt{\pi}} 2x\sin(x^2)\, dx = -\cos(x^2)\Big|_0^{\sqrt{\pi}}$$

$$= -\cos(\pi) - -\cos(0) = -(-1) + (1) = 2$$

2.) 

$$Vol = \int_0^2 \int_{x^2}^{2x} (x^2+y^2+10)\, dy\, dx - \int_0^2 \int_{x^2}^{2x} \left(2-\tfrac{2}{3}y-\tfrac{1}{3}x\right) dy\, dx$$

$$= \int_0^2 \left(x^2 y + \tfrac{1}{3}y^3 + 10y\right)\Big|_{y=x^2}^{y=2x} dx$$

$$- \int_0^2 \left(2y - \tfrac{1}{3}y^2 - \tfrac{1}{3}xy\right)\Big|_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 \left(2x^3 + \tfrac{8}{3}x^3 + 20x\right) - \left(x^4 + \tfrac{1}{3}x^6 + 10x^2\right) dx$$

$$- \int_0^2 \left(4x - \tfrac{4}{3}x^2 - \tfrac{2}{3}x^2\right) - \left(2x^2 - \tfrac{1}{3}x^4 - \tfrac{1}{3}x^3\right) dx$$

$$= \int_0^2 \left(\tfrac{-1}{3}x^6 - x^4 + \tfrac{14}{3}x^3 - 10x^2 + 20x\right) dx$$

$$- \int_0^2 \left(\tfrac{1}{3}x^4 + \tfrac{1}{3}x^3 - 4x^2 + 4x\right) dx$$

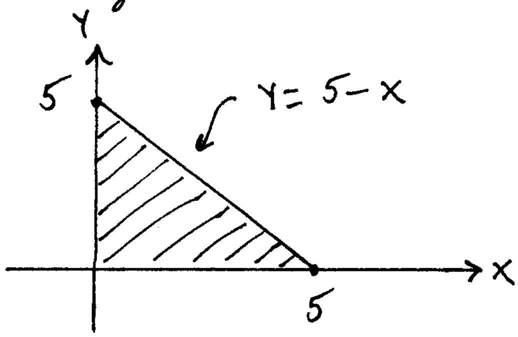
$$= \int_0^2 \left(\tfrac{-1}{3}x^6 - \tfrac{4}{3}x^4 + \tfrac{13}{3}x^3 - 6x^2 + 16x\right) dx$$

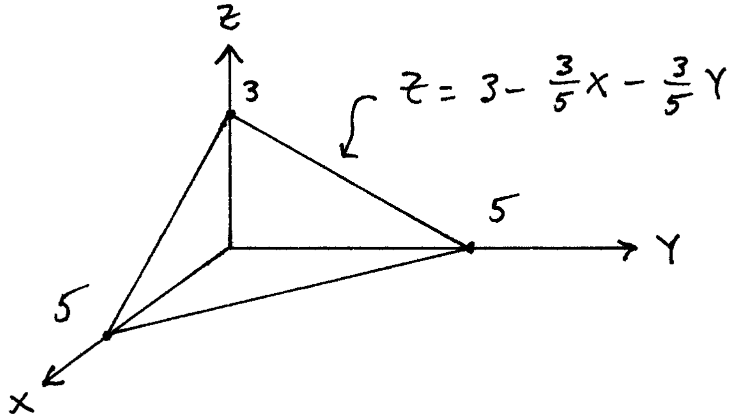
$$= \left(\tfrac{-1}{21}x^7 - \tfrac{4}{15}x^5 + \tfrac{13}{12}x^4 - 2x^3 + 8x^2\right)\Big|_0^2$$

$$= \frac{-128}{21} - \frac{128}{15} + \frac{208}{12} - 16 + 32 \quad \text{(calculator!)}$$

$$\approx \boxed{18.7}$$

3.) The intercepts for $3X + 3Y + 5Z = 15$ are $5, 5,$ and 3 so solid is a 3-sided block shown in diagram :





$$\text{Volume} = \int_0^5 \int_0^{5-x} \left(3 - \frac{3}{5}x - \frac{3}{5}y\right) dy\, dx$$

$$= \int_0^5 \left(3y - \frac{3}{5}xy - \frac{3}{10}y^2\right)\Big|_{y=0}^{y=5-x} dx$$

$$= \int_0^5 \left[3(5-x) - \frac{3}{5}x(5-x) - \frac{3}{10}(5-x)^2\right] dx$$

$$= \int_0^5 \left[15 - 3x - 3x + \frac{3}{5}x^2 - \frac{3}{10}(25 - 10x + x^2)\right] dx$$

$$= \int_0^5 \left[\frac{15}{2} - 3x + \frac{3}{10}x^2\right] dx$$

$$= \left(\frac{15}{2}x - \frac{3}{2}x^2 + \frac{1}{10}x^3\right)\Big|_0^5$$

$$= \frac{75}{2} - \frac{75}{2} + \frac{125}{10}$$

$$= \frac{25}{2}$$