

Section C.4

$$1.) \quad \frac{dy}{dx} = ky \rightarrow y = ce^{kx}$$

$$x=0, y=1 \rightarrow 1 = c \cdot 1 \rightarrow c=1 \rightarrow y = e^{kx}$$

$$x=3, y=2 \rightarrow 2 = e^{3k} \rightarrow \ln 2 = 3k \rightarrow$$

$$k = \frac{1}{3} \ln 2 = .23104906 \rightarrow$$

$$y = e^{.23104906x}$$

$$6.) \quad \frac{dy}{dx} = ky \rightarrow y = ce^{kx}$$

$$x=1, y=4 \rightarrow 4 = ce^k \rightarrow c = 4e^{-k}$$

$$x=2, y=1 \rightarrow 1 = ce^{2k} \rightarrow 1 = 4e^{-k} \cdot e^{2k} \rightarrow$$

$$\frac{1}{4} = e^k \rightarrow k = \ln \frac{1}{4} = -1.386294361 \text{ and}$$

$$c = 4e^{-k} = 16 \rightarrow$$

$$y = 16e^{-1.386294361x}$$

$$7.) \quad \frac{dA}{dt} = kA \rightarrow A = ce^{kt}; \text{ then}$$

$$\text{if } t=0, A = \$2000 \rightarrow 2000 = ce^0 = c \cdot 1 = c$$

$$\rightarrow A = 2000e^{kt}; \text{ if } t=5, A = \$2983.65$$

$$\text{then } 2983.65 = 2000e^{5k} \rightarrow$$

$$1.491825 = e^{5k} \rightarrow \ln 1.491825 = \ln e^{5k}$$

$$\rightarrow \ln 1.491825 = 5k \rightarrow$$

$k \approx 0.0800000405$ so that

$$A = 2000 e^{0.0800000405 t};$$

if $t = 10$ yrs, then $A \approx \$4451.08$.

15.)

$$\frac{dN}{dt} = kN(500 - N) \rightarrow \int \frac{1}{N(500 - N)} dN = \int k dt \rightarrow$$

$$\int \left[\frac{A}{N} + \frac{B}{500 - N} \right] dN = kt + c \rightarrow \int \left[\frac{\frac{1}{500}}{N} + \frac{\frac{1}{500}}{500 - N} \right] dN = kt + c \rightarrow$$

$$\frac{1}{500} \ln N - \frac{1}{500} \ln(500 - N) = kt + c \rightarrow$$

$$\ln N - \ln(500 - N) = 500kt + 500c \rightarrow$$

$$\ln \frac{N}{500 - N} = 500kt + c \rightarrow$$

$$\frac{N}{500 - N} = e^{500kt + c} = e^c e^{500kt} = ce^{500kt} \rightarrow$$

$$\frac{N}{500 - N} = ce^{500kt}; \text{ if } \underline{t=0}, \underline{N=100} \text{ then}$$

$$\frac{100}{400} = c \cdot e^0 \rightarrow c = \frac{1}{4} \rightarrow \frac{N}{500 - N} = \frac{1}{4} e^{500kt};$$

if $\underline{t=4}, \underline{N=200}$ then

$$\frac{200}{300} = \frac{1}{4} e^{2000k} \rightarrow \frac{8}{3} = e^{2000k} \rightarrow$$

$$\ln \frac{8}{3} = \ln e^{2000k} = 2000k \rightarrow$$

$$k = \frac{1}{2000} \ln \frac{8}{3} = 0.000490414 \text{ (use later)} \rightarrow$$

$$\frac{N}{500-N} = \frac{1}{4} e^{500kt} \rightarrow$$

$$N = 125 e^{500kt} - \frac{1}{4} e^{500kt} \cdot N \rightarrow$$

$$N + \frac{1}{4} e^{500kt} N = 125 e^{500kt} \rightarrow$$

$$\left(1 + \frac{1}{4} e^{500kt}\right) N = 125 e^{500kt} \rightarrow$$

$$N = \frac{125 e^{500kt}}{1 + \frac{1}{4} e^{500kt}} \rightarrow$$

$$N = \frac{500}{4e^{-500kt} + 1} \rightarrow \text{(plug in } k)$$

$$N = \frac{500}{4e^{-0.245207313t} + 1}.$$

20.) $y = \frac{-1}{kt+c}$ and

$$t=0, y=75 \rightarrow 75 = \frac{-1}{c} \rightarrow c = \frac{-1}{75} \text{ and}$$

$$t=1, y=12 \rightarrow 12 = \frac{-1}{k + \frac{-1}{75}} = \frac{-75}{75k-1} \rightarrow$$

$$12(75k-1) = -75 \rightarrow 900k = -63 \rightarrow$$

$$k = -.07 \rightarrow y = \frac{-1}{-.07t + \frac{-1}{75}} \quad \text{or}$$

$$y = \frac{75}{5.25t + 1}$$

$$28.) \quad \frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= \left(0 \frac{\text{lbs.}}{\text{gal}}\right) \left(\frac{5 \text{ gal.}}{\text{min.}}\right) - \left(\frac{Q \text{ lbs.}}{100 \text{ gal.}}\right) \left(\frac{5 \text{ gal.}}{\text{min.}}\right)$$

$$\rightarrow \frac{dQ}{dt} = -\frac{Q}{20} \rightarrow \frac{dQ}{dt} = \left(\frac{-1}{20}\right) Q$$

(rate is proportional to amount!)

$$\rightarrow Q = c e^{\frac{-1}{20}t}; \quad \text{if } t=0, Q=25 \text{ lbs.}$$

$$\text{then } 25 = c e^0 = c \cdot 1 = c \rightarrow$$

$$a.) \quad Q = 25 e^{\frac{-1}{20}t}$$

$$b.) \quad \text{If } Q=15 \text{ lbs., then } 15 = 25 e^{\frac{-1}{20}t} \rightarrow$$

$$\frac{3}{5} = e^{\frac{-1}{20}t} \rightarrow \ln\left(\frac{3}{5}\right) = \ln e^{\frac{-1}{20}t} \rightarrow$$

$$\ln\left(\frac{3}{5}\right) = \frac{-1}{20}t \rightarrow$$

$$t = -20 \cdot \ln\left(\frac{3}{5}\right) \approx 10.21 \text{ min.}$$

$$29.) \quad \frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= \left(\frac{\frac{1}{2} \text{ lb.}}{\text{gal.}} \right) \left(\frac{5 \text{ gal.}}{\text{min.}} \right) - \left(\frac{Q \text{ lbs.}}{100 \text{ gal.}} \right) \left(\frac{5 \text{ gal.}}{\text{min.}} \right) \rightarrow$$

$$\frac{dQ}{dt} = \frac{5}{2} - \frac{1}{20} Q \rightarrow \frac{dQ}{dt} + \frac{1}{20} Q = \frac{5}{2} \rightarrow$$

(first-order linear) $\mu = e^{\int \frac{1}{20} dt} = e^{\frac{1}{20} t} \rightarrow$

$$e^{\frac{1}{20} t} \cdot \frac{dQ}{dt} + \frac{1}{20} e^{\frac{1}{20} t} \cdot Q = \frac{5}{2} e^{\frac{1}{20} t} \rightarrow$$

$$D(e^{\frac{1}{20} t} \cdot Q) = \frac{5}{2} e^{\frac{1}{20} t} \rightarrow$$

$$e^{\frac{1}{20} t} \cdot Q = \int \frac{5}{2} e^{\frac{1}{20} t} dt = \frac{5}{2} \cdot 20 e^{\frac{1}{20} t} + c \rightarrow$$

$$e^{\frac{1}{20} t} \cdot Q = 50 e^{\frac{1}{20} t} + c \rightarrow \underline{Q = 50 + c e^{-\frac{1}{20} t}} ;$$

$$t=0, Q=0 \rightarrow 0 = 50 + c e^0 = 50 + c \rightarrow c = -50$$

$$\rightarrow \boxed{Q = 50 - 50 e^{-\frac{1}{20} t}} ;$$

if $t = 30$ min, then

$$Q = 50 - 50 e^{-\frac{1}{20}(30)} \approx \underline{38.84 \text{ lbs.}}$$

$$30.) \quad \frac{ds}{dh} = \frac{k}{h} \rightarrow \int ds = \int \frac{k}{h} dh \rightarrow$$

$$S = k \ln h + C$$

and

$$h=2, s=25 \rightarrow$$

$$25 = k \ln 2 + C$$

$$h=6, s=12 \rightarrow$$

$$12 = k \ln 6 + C$$

$$\text{(subtract)} \quad 13 = k \ln 2 - k \ln 6 \rightarrow$$

$$13 = k (\ln 2 - \ln 6) = k \ln \frac{2}{6} = k \ln \frac{1}{3} \rightarrow$$

$$k = \frac{13}{\ln \frac{1}{3}} = \underline{-11.83310995} \quad ; \text{ so}$$

$$C = 25 - k \ln 2 = 33.2020868 \rightarrow$$

$$S \approx -11.83310995 \ln h + 33.2020868$$

31.) Let M be amount of moisture (given in ounces, for example) at time t ; then

$$\frac{dM}{dt} = kM \rightarrow M = c e^{kt}$$

\uparrow initial amount of moisture;

if $t=1$, then $M = 60\%$ of $c = 0.6c \rightarrow$

$$0.6c = c e^k \rightarrow 0.6 = e^k \rightarrow \ln(0.6) = \ln e^k$$

$$\rightarrow k = \ln(0.6) \rightarrow M = c e^{(\ln 0.6)t} \rightarrow$$

$$M = c (e^{\ln(0.6)})^t = c (0.6)^t \rightarrow \boxed{M = c (0.6)^t};$$

if $M = 20\%$ of $c = 0.2c$, then

$$0.2\phi = \phi(0.6)^t \rightarrow \ln(0.2) = \ln(0.6)^t \rightarrow$$

$$\ln(0.2) = t \cdot \ln(0.6) \rightarrow$$

$$t = \frac{\ln(0.2)}{\ln(0.6)} \approx \text{3.15 hrs.}$$

$$32.) \quad \frac{1}{x} \frac{dx}{dt} = \frac{1}{y} \frac{dy}{dt} \rightarrow \int \frac{1}{x} dx = \int \frac{1}{y} dy \rightarrow$$

$$\ln x = \ln y + c \rightarrow e^{\ln x} = e^{\ln y + c} \rightarrow$$

$$x = e^c e^{\ln y} = cy \rightarrow x = cy \rightarrow y = \frac{1}{c}x$$

$$\rightarrow \text{y = cx}$$

$$33.) \quad \frac{dP}{dt} = kP + N \rightarrow P' - kP = N$$

(this is first-order linear!)

$$e^{\int -k dt} = e^{-kt} \rightarrow$$

$$e^{-kt} P' - k e^{-kt} P = N e^{-kt} \rightarrow$$

$$D(e^{-kt} P) = N e^{-kt} \rightarrow$$

$$e^{-kt} P = \int N e^{-kt} dt = N \cdot \frac{1}{-k} e^{-kt} + c \rightarrow$$

$$P = \frac{-N}{k} + c e^{kt}$$

Q3: Determine the maximum value

of $S = 1000 - 4t - 0.000174413 (500 - 2t)^{5/2}$:

$$S' = -4 + 0.000872065 (500 - 2t)^{3/2} = 0 \rightarrow$$

$$(500 - 2t)^{3/2} = 4586.81 \rightarrow$$

$$500 - 2t = 4586.81^{2/3} \rightarrow$$

$$500 - 2t = 276.1 \rightarrow$$

$$t = 112 \text{ min.} \quad \text{and}$$

maximum amount is

$$S = 331.3 \text{ lbs.}$$

Math 16C
Kouba
Worksheet 1

Let S represent the amount (in pounds) of salt in each tank at time t minutes. Find a formula for S for each of the following and then answer the particular questions.

1.) A solution containing $1/2$ lb. of salt per gallon flows into a tank at the rate of 2 gal./min. and the well-stirred mixture flows out of the tank at the same rate. The tank initially holds 100 gallons of solution containing 5 lbs. of salt.

- a.) How much salt is in the tank after 30 minutes ?
- b.) How much salt do you expect to be in the tank as t gets infinitely large ?

2.) A solution containing 1 lb. of salt per gallon flows into a tank at the rate of 5 gal./min. and the well-stirred mixture flows out of the tank at the rate of 4 gal./min. The tank initially holds 50 gallons of water containing no salt.

- a.) How many gallons of solution are in the tank after 1 hour ?
- b.) How much salt is in the tank after 1 hour ?
- c.) Assuming that the tank is very large, how much salt per gallon do you expect to be in the tank as t get infinitely large ?

Worksheet 1

$$\begin{aligned} \textcircled{1} \quad \frac{dS}{dt} &= (\text{rate in}) - (\text{rate out}) \\ &= \left(\frac{1}{2} \frac{\text{lb.}}{\text{gal.}}\right) \left(2 \frac{\text{gal.}}{\text{min.}}\right) - \left(\frac{S \text{ lb.}}{100 \text{ gal.}}\right) \left(2 \frac{\text{gal.}}{\text{min.}}\right) \\ &= 1 - \frac{1}{50} S \quad \rightarrow \quad \frac{dS}{dt} + \frac{1}{50} S = 1 \quad \rightarrow \end{aligned}$$

$$e^{\int \frac{1}{50} dt} = e^{\frac{1}{50} t} \rightarrow e^{\frac{1}{50} t} S' + \frac{1}{50} e^{\frac{1}{50} t} S = e^{\frac{1}{50} t} \rightarrow$$

$$D(e^{\frac{1}{50} t} S) = e^{\frac{1}{50} t} \rightarrow e^{\frac{1}{50} t} S = \int e^{\frac{1}{50} t} dt \rightarrow$$

$$e^{\frac{1}{50} t} S = 50 e^{\frac{1}{50} t} + c \quad \rightarrow$$

$$\boxed{S = 50 + c e^{-\frac{1}{50} t}} \quad ; \quad \text{and}$$

$$t=0, S=5 \text{ lbs.} \rightarrow 5 = 50 + c e^0 \rightarrow c = -45 \rightarrow$$

$$\boxed{S = 50 - 45 e^{-\frac{1}{50} t}}$$

$$\textcircled{a} \quad t = 30 \text{ min.} \rightarrow S = \boxed{25.3 \text{ lbs.}}$$

$$\textcircled{b} \quad \lim_{t \rightarrow \infty} S = 50 - 45 \cdot (0) = \boxed{50 \text{ lbs.}}$$

$$\begin{aligned}
 \textcircled{2} \quad \frac{dS}{dt} &= (\text{rate in}) - (\text{rate out}) \\
 &= \left(1 \frac{\text{lb.}}{\text{gal.}}\right) \left(5 \frac{\text{gal.}}{\text{min.}}\right) - \left(\frac{S \text{ lb.}}{50+t \text{ gal.}}\right) \left(4 \frac{\text{gal.}}{\text{min.}}\right) \\
 &= 5 - \frac{4}{50+t} S \quad \text{or}
 \end{aligned}$$

$$\frac{dS}{dt} + \frac{4}{50+t} S = 5; \quad \text{then}$$

$$\begin{aligned}
 e^{\int \frac{4}{50+t} dt} &= e^{4 \ln(50+t)} = e^{\ln(50+t)^4} \\
 &= (50+t)^4 \rightarrow
 \end{aligned}$$

$$(50+t)^4 S' + 4(50+t)^3 S = 5(50+t)^4 \rightarrow$$

$$D \left\{ (50+t)^4 S \right\} = 5(50+t)^4 \rightarrow$$

$$(50+t)^4 S = \int 5(50+t)^4 dt = (50+t)^5 + C \rightarrow$$

$$S = (50+t) + \frac{C}{(50+t)^4}; \quad \text{and}$$

$$t=0, S=0 \text{ lb.} \rightarrow$$

$$0 = 50 + \frac{C}{50^4} \rightarrow C = -50^5 \rightarrow$$

$$S = 50+t - \frac{50^5}{(50+t)^4}$$

a.) $50 + 60 = 110 \text{ gal.}$

b.) $t = 60 \text{ min.} \rightarrow S = 107.87 \text{ lbs.}$

c.) salt per gallon is given by

$$\lim_{t \rightarrow \infty} \frac{50 + t - \frac{50^5}{(50+t)^4}}{50+t} \quad \begin{array}{l} \text{lbs.} \\ \text{gal.} \end{array}$$

$$= \lim_{t \rightarrow \infty} \left[1 - \frac{50^5}{(50+t)^5} \right] = 1 \text{ lb./gal.}$$