

Section 7.5

2.) $z = x^2 + y^2 + 2x - 6y + 6 \rightarrow$

$$z_x = 2x + 2 = 0 \rightarrow \boxed{x = -1},$$

$$z_y = 2y - 6 = 0 \rightarrow \boxed{y = 3}; \text{ then}$$

$$z_{xx} = 2, \quad z_{yy} = 2, \quad z_{xy} = 0 \text{ and}$$

For $\boxed{(-1, 3)}$: $D = z_{xx}z_{yy} - (z_{xy})^2 = (2)(2) - (0)^2 = 4 > 0$

and $z_{xx} = 2 > 0$ so $(-1, 3)$ determines a minimum value of $\boxed{z = -4}$.

3.) $z = \sqrt{x^2 + y^2 + 1} \rightarrow$

$$z_x = \frac{1}{2}(x^2 + y^2 + 1)^{-1/2} \cdot (2x) = \frac{x}{(x^2 + y^2 + 1)^{1/2}} = 0 \rightarrow$$

$$\boxed{x = 0}; \quad z_y = \frac{1}{2}(x^2 + y^2 + 1)^{-1/2} \cdot (2y) = \frac{y}{(x^2 + y^2 + 1)^{1/2}} = 0 \rightarrow$$

$$\boxed{y = 0};$$

$$z_{xx} = \frac{(x^2 + y^2 + 1)^{1/2}(1) - x \cdot \frac{1}{2}(x^2 + y^2 + 1)^{-1/2}(2x)}{((x^2 + y^2 + 1)^{1/2})^2},$$

$$z_{yy} = \frac{(x^2 + y^2 + 1)^{1/2}(1) - y \cdot \frac{1}{2}(x^2 + y^2 + 1)^{-1/2}(2y)}{((x^2 + y^2 + 1)^{1/2})^2},$$

$$z_{xy} = \frac{(x^2 + y^2 + 1)^{1/2}(0) - x \cdot \frac{1}{2}(x^2 + y^2 + 1)^{-1/2} \cdot 2y}{((x^2 + y^2 + 1)^{1/2})^2};$$

$$z_{xx}(0,0) = \frac{1-0}{1} = 1, \quad z_{yy}(0,0) = \frac{1-0}{1} = 1,$$

$$z_{xy}(0,0) = \frac{0}{1} = 0, \text{ then}$$

For $\boxed{(0,0)}$: $D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (1)(1) - (0)^2 = 1 > 0$

and $z_{xx} = 1 > 0$ so $(0,0)$ determines a minimum value of $\boxed{z = 1}$.

$$5.) \quad z = (x-1)^2 + (y-3)^2 \quad \rightarrow$$

$$z_x = 2(x-1) = 0 \rightarrow \boxed{x=1} \quad \left. \vphantom{z_x} \right\} \boxed{(1,3)}$$

$$z_y = 2(y-3) = 0 \rightarrow \boxed{y=3}$$

$$z_{xx} = 2, \quad z_{yy} = 2, \quad z_{xy} = 0 \quad \rightarrow$$

For $\boxed{(1,3)}$:

$$D = z_{xx} z_{yy} - (z_{xy})^2 = (2)(2) - (0)^2 = 4 > 0$$

and $z_{xx} = 2 > 0$ so $(1,3)$ determines a minimum value of $z = 0$.

$$7.) \quad z = 2x^2 + 2xy + y^2 + 2x - 3 \quad \rightarrow$$

$$z_x = 4x + 2y + 2 = 0 \rightarrow \boxed{y = -2x - 1} \quad \left. \vphantom{z_x} \right\}$$

$$z_y = 2x + 2y = 0 \rightarrow \boxed{y = -x}$$

$$\rightarrow -x = -2x - 1 \rightarrow \boxed{x = -1} \text{ and } \boxed{y = 1} \rightarrow \boxed{(-1,1)} :$$

$$z_{xx} = 4, \quad z_{yy} = 2, \quad z_{xy} = 2 \quad \text{so}$$

For $\boxed{(-1,1)}$:

$$D = z_{xx} z_{yy} - (z_{xy})^2 = (4)(2) - (2)^2 = 4 > 0$$

and $z_{xx} = 4 > 0$ so $(-1,1)$ determines a minimum value of $z = -4$.

$$9.) \quad z = -5x^2 + 4xy - y^2 + 16x + 10 \quad \rightarrow$$

$$z_x = -10x + 4y + 16 = 0 \rightarrow y = -4 + \frac{5}{2}x \quad \left. \vphantom{z_x} \right\}$$

$$z_y = 4x - 2y = 0 \rightarrow y = 2x$$

$$2x = -4 + \frac{5}{2}x \rightarrow 4 = \frac{1}{2}x \rightarrow \boxed{x = 8} \text{ and } \boxed{y = 16}$$

For $\boxed{(8,16)}$:

$$z_{xx} = -10, \quad z_{yy} = -2, \quad z_{xy} = 4 \quad \rightarrow$$

For $(8, 16)$:

$$D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (-10)(-2) - (4)^2 = 4 > 0 \text{ and}$$
$$z_{xx} = -10 \text{ so } (8, 16) \text{ determines a}$$

maximum value of $(z = 74)$.

10.) $z = x^2 + 6xy + 10y^2 - 4y + 4 \rightarrow$

$$z_x = 2x + 6y = 2(x + 3y) = 0 \rightarrow \boxed{x + 3y = 0},$$

$$z_y = 6x + 20y - 4 = 2(3x + 10y - 2) = 0 \rightarrow$$

$$\boxed{3x + 10y - 2 = 0}; \quad x = -3y \xrightarrow{\text{sub}}$$

$$3(-3y) + 10y - 2 = 0 \rightarrow -9y + 10y - 2 = 0 \rightarrow$$

$$\boxed{y = 2} \text{ and } \boxed{x = -6};$$

$$z_{xx} = 2, \quad z_{yy} = 20, \quad z_{xy} = 6 \text{ then}$$

For $(-6, 2)$: $D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (2)(20) - (6)^2 = 4 > 0$

and $z_{xx} = 2 > 0$ so $(-6, 2)$ determines a
minimum value of $(z = 0)$

14.) $z = x^2 - 3xy - y^2 \rightarrow$

$$z_x = 2x - 3y = 0 \rightarrow \boxed{x = \frac{3}{2}y},$$

$$z_y = -3x - 2y = 0 \rightarrow \boxed{x = -\frac{2}{3}y}; \text{ then}$$

$$\frac{3}{2}y = -\frac{2}{3}y \rightarrow \frac{5}{6}y = 0 \rightarrow \boxed{y = 0, x = 0},$$

$$z_{xx} = 2, \quad z_{yy} = -2, \quad z_{xy} = -3$$

For $(0, 0)$: $D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (2)(-2) - (-3)^2 = -13 < 0$
so $(0, 0)$ determines a saddle
point at $z = 0$.

$$18.) \quad z = e^{-(x^2+y^2)} \quad \rightarrow$$

$$\left. \begin{aligned} z_x &= -2x e^{-(x^2+y^2)} = 0 \rightarrow x=0 \\ z_y &= -2y e^{-(x^2+y^2)} = 0 \rightarrow y=0 \end{aligned} \right\} \text{so critical point is } \boxed{(0,0)};$$

$$z_{xx} = 4x^2 e^{-(x^2+y^2)} - 2e^{-(x^2+y^2)} = (4x^2 - 2)e^{-(x^2+y^2)}$$

$$z_{yy} = 4y^2 e^{-(x^2+y^2)} - 2e^{-(x^2+y^2)} = (4y^2 - 2)e^{-(x^2+y^2)},$$

$$z_{xy} = 4xy e^{-(x^2+y^2)} \quad \rightarrow$$

For (0,0): $D = z_{xx} z_{yy} - (z_{xy})^2 = (-2)(-2) - (0)^2 = 4 > 0$

and $z_{xx} = -2 < 0$ so $(0,0)$ determines a maximum value of $z = 1$.

$$20.) \quad z = \frac{-4x}{x^2+y^2+1} \quad \rightarrow$$

$$z_x = \frac{(x^2+y^2+1)(-4) - (-4x)(2x)}{(x^2+y^2+1)^2}$$

$$= \frac{-4x^2 - 4y^2 - 4 + 8x^2}{(x^2+y^2+1)^2} = \frac{4x^2 - 4y^2 - 4}{(x^2+y^2+1)^2}$$

$$z_y = \frac{(x^2+y^2+1)(0) - (-4x)(2y)}{(x^2+y^2+1)^2} = \frac{8xy}{(x^2+y^2+1)^2}$$

$$z_{xx} = \frac{(x^2+y^2+1)^2(8x) - (4x^2 - 4y^2 - 4) \cdot 2(x^2+y^2+1)(2x)}{(x^2+y^2+1)^4}$$

$$= \frac{8x(x^2+y^2+1) \cdot [(x^2+y^2+1) - (2x^2 - 2y^2 - 2)]}{(x^2+y^2+1)^4}$$

$$= \frac{8x(3-x^2+3y^2)}{(x^2+y^2+1)^3}$$

$$z_{yy} = \frac{(x^2+y^2+1)^2(8x) - 8xy \cdot 2(x^2+y^2+1) \cdot 2y}{(x^2+y^2+1)^4}$$

$$= \frac{8x(x^2+y^2+1)[(x^2+y^2+1) - 4y^2]}{(x^2+y^2+1)^4}$$

$$= \frac{8x(1+x^2-3y^2)}{(x^2+y^2+1)^3}$$

$$z_{xy} = z_{yx} = \frac{(x^2+y^2+1)^2(8y) - 8xy \cdot 2(x^2+y^2+1) \cdot 2x}{(x^2+y^2+1)^4}$$

$$= \frac{8y(x^2+y^2+1)[(x^2+y^2+1) - 4x^2]}{(x^2+y^2+1)^4}$$

$$= \frac{8y(1+y^2-3x^2)}{(x^2+y^2+1)^3}$$

critical points:

$$z_x = 0 \rightarrow 4x^2 - 4y^2 - 4 = 0 \rightarrow \boxed{x^2 - y^2 - 1 = 0};$$

$$z_y = 0 \rightarrow 8xy = 0 \rightarrow \boxed{x=0 \text{ or } y=0}; \text{ then}$$

$$x=0 \rightarrow -y^2 - 1 = 0 \rightarrow -(y^2 + 1) = 0 \text{ (impossible)};$$

$$y=0 \rightarrow x^2 - 1 = (x-1)(x+1) = 0 \rightarrow x=1 \text{ or } x=-1$$

so critical pts. are $\boxed{x=1, y=0}$ and $\boxed{x=-1, y=0}$;

$$\text{For } \underline{(1,0)}: D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (2)(2) - (0)^2 = 4 > 0$$

and $z_{xx} > 0$ so $(1,0)$ determines a

minimum value of $\boxed{z = -2}$;

$$\text{For } \underline{(-1,0)}: D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (-2)(-2) - (0)^2 = 4 > 0$$

and $z_{xx} < 0$ so $(-1, 0)$ determines a
maximum value of $z = 2$

Math 16C
Kouba
Worksheet 4

1.) Find z_x and z_y for each of the following functions.

a.) $f(x, y) = \sin 3x + \cos 5y$

b.) $f(x, y) = \cos(x^2y)$

c.) $f(x, y) = \ln(\sec x - \tan y)$

d.) $f(x, y) = e^{x^2 \sin y}$

e.) $f(x, y) = \frac{\ln y}{\ln x + \ln y}$

f.) $f(x, y) = \cot^2(4y) - \csc^3(5xy)$

g.) $f(x, y) = \sqrt{7y^2 + 6 \cot(11 - \pi e^{(-1/2)\sqrt{y}})}$

h.) $f(x, y) = \ln\{\ln\{\ln(3x - 9y^2)\}\}$

2.) Find $z_x, z_y, z_{xx}, z_{yy},$ and z_{xy} for $f(x, y) = \sin(xy)$.

3.) Let $f(x, y) = (y + 7x)^2$. Show that $z_{xx} = 49 \cdot z_{yy}$.

4.) Find a function $z = f(x, y)$ so that

$$z_x = y^3 - 2x + 5 \quad \text{and} \quad z_y = 2ye^{y^2} + 3xy^2 .$$

Worksheet 4

1.

a.) $z = \sin 3x + \cos 5y \rightarrow$

$$z_x = \cos 3x \cdot 3, \quad z_y = -\sin 5y \cdot 5$$

b.) $z = \cos(x^2 y) \rightarrow$

$$z_x = -\sin(x^2 y) \cdot 2xy,$$

$$z_y = -\sin(x^2 y) \cdot x^2$$

c.) $z = \ln(\sec x - \tan y) \rightarrow$

$$z_x = \frac{1}{\sec x - \tan y} \cdot \sec x \tan x,$$

$$z_y = \frac{1}{\sec x - \tan y} \cdot -\sec^2 y$$

d.) $z = e^{x^2 \sin y} \rightarrow$

$$z_x = e^{x^2 \sin y} \cdot 2x \sin y,$$

$$z_y = e^{x^2 \sin y} \cdot x^2 \cos y$$

e.) $z = \frac{\ln y}{\ln x + \ln y} \rightarrow$

$$z_x = \frac{(\ln x + \ln y)(0) - \ln y \left(\frac{1}{x}\right)}{(\ln x + \ln y)^2},$$

$$z_y = \frac{(\ln x + \ln y)\left(\frac{1}{y}\right) - \ln y \left(\frac{1}{y}\right)}{(\ln x + \ln y)^2}$$

$$f.) \quad z = \cot^2(4Y) - \csc^3(5XY) \rightarrow$$

$$z_x = 0 - 3 \csc^2(5XY) \cdot (-1) \csc(5XY) \cot(5XY) \cdot \{5Y\}$$

$$z_y = 2 \cot(4Y) \cdot (-1) \csc^2(4Y) \cdot \{4\} - 3 \csc^2(5XY) \cdot (-1) \csc(5XY) \cot(5XY) \cdot \{5X\}$$

$$g.) \quad z = \left\{ 7Y^2 + 6 \cot \left(11 - \pi e^{-\frac{1}{2}Y^{\frac{1}{2}}} \right) \right\}^{\frac{1}{2}} \rightarrow z_x = 0!$$

$$z_y = \frac{1}{2} \left\{ 7Y^2 + 6 \cot \left(11 - \pi e^{-\frac{1}{2}Y^{\frac{1}{2}}} \right) \right\}^{-\frac{1}{2}} \cdot [14Y +$$

$$\left. -6 \csc^2 \left(11 - \pi e^{-\frac{1}{2}Y^{\frac{1}{2}}} \right) \cdot \left\{ -\pi \cdot e^{-\frac{1}{2}Y^{\frac{1}{2}}} \cdot \frac{-1}{2} \cdot \frac{1}{2} Y^{-\frac{1}{2}} \right\} \right]$$

$$h.) \quad z = \ln(\ln(\ln(3X - 9Y^2))) \rightarrow$$

$$z_x = \frac{1}{\ln(\ln(3X - 9Y^2))} \cdot \frac{1}{\ln(3X - 9Y^2)} \cdot \frac{1}{3X - 9Y^2} \cdot (3)$$

$$z_y = \frac{1}{\ln(\ln(3X - 9Y^2))} \cdot \frac{1}{\ln(3X - 9Y^2)} \cdot \frac{1}{3X - 9Y^2} \cdot (-18Y)$$

$$2.) \quad z = \sin(XY) \rightarrow$$

$$z_x = \cos(XY) \cdot Y$$

$$z_y = \cos(XY) \cdot X$$

$$z_{xx} = -Y \sin(XY) \cdot Y$$

$$z_{yy} = -X \sin(XY) \cdot X$$

$$z_{xy} = \cos(XY) \cdot 1 + Y \cdot -\sin(XY) \cdot X$$

$$z_{yx} = \cos(XY) \cdot 1 + X \cdot -\sin(XY) \cdot Y$$

$$3.) \quad z = (y+7x)^2 \rightarrow$$

$$z_x = 2(y+7x) \cdot 7 = 14y + 98x, \quad z_{xx} = 98 \quad \text{and}$$
$$z_y = 2(y+7x) \cdot 1 = 2y + 14x, \quad z_{yy} = 2 \quad \text{then}$$

$$z_{xx} = 98 = 49 \cdot 2 = 49 \cdot z_{yy}$$

4.) $z_x = y^3 - 2x + 5$ so "partial anti-derivative" is

$$z = xy^3 - x^2 + 5x + c(y), \quad \text{where}$$

$c(y)$ represents a function of y .

Now take the y -partial derivative of z getting

$$z_y = 3xy^2 + c'(y) = 3xy^2 + 2ye^{y^2}$$

$$\text{Then } c'(y) = 2ye^{y^2} \rightarrow$$

$$c(y) = e^{y^2} + c_1, \quad \text{where } c_1 \text{ is}$$

any constant. Thus,

$$\boxed{z = xy^3 - x^2 + 5x + e^{y^2} + c_1}$$