KEY

Please PRINT your name here: __________________________

Your Exam ID Number ________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. COPYING ANSWERS FROM A CLASSMATE's EXAM IS A VIOLATION OF THE UNIVERSITY HONOR CODE.

4. HAVING ANOTHER PERSON TAKE YOUR EXAM FOR YOU IS A VIOLATION OF THE UNIVERSITY HONOR CODE.

5. No notes, books, or handouts may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

6. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

7. You will be graded on proper use of derivative notation.

8. Put units on answers where units are appropriate.

9. Make sure that you have 8 pages, including the cover page.

10. You have until ________ sharp to finish the exam.
1.) (12 pts.) Compute \( z_x, z_y, \) and \( z_{xy} \) for \( z = x^3 + \sin y + e^{xy} \).

\[
\begin{align*}
z_x &= 3x^2 + ye^{xy} \\
zz &= \cos y + xe^{xy} \\
zz &= \frac{\partial}{\partial y} (3x^2 + ye^{xy}) \\
zz &= yxe^{xy} + 1 \cdot e^{xy}
\end{align*}
\]

2.) (12 pts.) Evaluate the double integral \( \int_0^1 \int_0^x (2x - 3y^2) \, dy \, dx \).

\[
\begin{align*}
&= \int_0^1 \left. (2xy - y^3) \right|_0^x \, dx \\
&= \int_0^1 (2x^2 - x^3) \, dx \\
&= \left. \left( \frac{2}{3}x^3 - \frac{1}{4}x^4 \right) \right|_0^1 \\
&= \frac{2}{3} - \frac{1}{4} \\
&= \frac{5}{12}
\end{align*}
\]
3.) (12 pts.) Evaluate the double integral \( \int_{0}^{2} \int_{y/2}^{1} e^{-x^2} \, dx \, dy \). (HINT: Switch the order of integration.)

\[
\begin{align*}
\int_{0}^{2} \int_{y/2}^{1} e^{-x^2} \, dx \, dy &= \int_{y=0}^{y=2} \int_{x=y/2}^{1} e^{-x^2} \, dx \, dy \\
&= \int_{y=0}^{y=2} \left[ \int_{x=y/2}^{1} e^{-x^2} \, dx \right] \, dy \\
&= \int_{0}^{2} 2xe^{-x^2} \, dx \\
&= -e^{-x^2} \bigg|_{0}^{1} \\
&= -e^{-1} + e^{0} \\
&= 1 - \frac{1}{e} 
\end{align*}
\]

4.) (6 pts. each) Consider the function \( f(x, y) = xy \) defined on the region \( R \) given in the diagram. SET BUT DO NOT EVALUATE integral(s) for each of the following.

a.) the average width of region \( R \)

\[
\text{width } w(y) = y^{1/3} - \frac{1}{4} y, \quad \text{so} \\
\text{AVE} = \frac{1}{8-0} \int_{0}^{8} (y^{1/3} - \frac{1}{4} y) \, dy 
\]

b.) the average value of \( f(x, y) \) on region \( R \)

\[
\text{Area } R = \int_{0}^{2} \int_{x^3}^{4x} 1 \, dy \, dx, \quad \text{so} \\
\text{AVE} = \frac{1}{\text{Area } R} \int_{0}^{2} \int_{x^3}^{4x} xy \, dy \, dx 
\]
5.) (12 pts.) Find and classify each critical point as that which determines a relative maximum value, a relative minimum value, or a saddle point for the function

\[ f(x, y) = x^3 - y^3 + 3xy \]

\[
\frac{\partial f}{\partial x} = 3x^2 + 3y = 0 \rightarrow y = -x^2 \quad (\text{sub})
\]

\[
\frac{\partial f}{\partial y} = -3y^2 + 3x = 0 \rightarrow x = y^2
\]

\[
\rightarrow x = (-x^2)^2 = x^4 \rightarrow x^4 - x = 0 \rightarrow
\]

\[ x(x^3 - 1) = 0 \rightarrow x = 0, y = 0 \quad \text{or} \quad x = 1, y = -1 \]

\[ z_{xx} = 6x, \quad z_{yy} = -6y, \quad z_{xy} = 3 \]

**Test (0,0):**

\[
D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (6)(0) - (3)^2 = -9 < 0
\]

so (0,0) determines a saddle point with \( z = 0 \).

**Test (1, -1):**

\[
D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (6)(6) - (3)^2 = 27 > 0
\]

and \( z_{xx} = 6 > 0 \) so (1, -1) determines a minimum value of \( z = -1 \).
6.) (10 pts.) Use the Method of Lagrange Multipliers to minimize \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to the constraint \( x - 2y + 6z = 82 \).

\[
F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda (x - 2y + 6z - 82) \\
= x^2 + y^2 + z^2 - \lambda x + 2\lambda y - 6\lambda z + 82\lambda \\
\Rightarrow \\
\begin{aligned}
F_x &= 2x - \lambda = 0 \Rightarrow \lambda = 2x \\
F_y &= 2y + 2\lambda = 0 \Rightarrow y = -\lambda \\
F_z &= 2z - 6\lambda = 0 \Rightarrow z = 3\lambda \\
F_\lambda &= -x + 2y - 6z + 82 = 0
\end{aligned}
\]

then
\[
\begin{aligned}
-x + 2(-2x) - 6(6x) + 82 &= 0 \\
-x - 4x - 36x &= -82 \\
-41x &= -82 \\
x &= 2
\end{aligned}
\]

\( y = -4 \), \( z = 12 \), \( \lambda = 4 \), and

minimum value

\[
f(2, -4, 12) = 2^2 + (-4)^2 + (12)^2 = 164
\]
7.) (10 pts.) Use the Method of Lagrange Multipliers to minimize \( S = x^2 + y^2 \) subject to the two constraints \( z = y + 3 \) and \( x - y = 2 \)

Let \( F(x, y, z, \lambda, \mu) = (x^2 + y^2) \)
\[-\lambda (z - y - 3) - \mu (x - y - 2) \rightarrow \]

\[
\begin{align*}
F_x &= 2x - \mu = 0 \quad \rightarrow \quad \mu = 2x \\
F_y &= 2y + \lambda + \mu = 0 \quad \rightarrow \quad \mu = -2y \\
F_z &= -\lambda = 0 \quad \rightarrow \quad \lambda = 0 \\
\end{align*}
\]

\[2x = -2y \]

\[
\begin{cases}
F_{\lambda} = -z + y + 3 = 0 \\
F_{\mu} = -x + y + 2 = 0 \quad \rightarrow \quad -x + (-x) + 2 = 0
\end{cases}
\]

\[-2x = -2 \quad \rightarrow \quad x = 1 \quad \rightarrow \quad y = -1 \quad \rightarrow \]

\[z = y + 3 = (-1) + 3 \quad \rightarrow \quad z = 2 \quad \text{and} \]

minimum \( S = (1)^2 + (-1)^2 \quad \rightarrow \quad S = 2 \)
8.) a.) (5 pts.) Determine the domain of \( f(x, y) = 3 + \sqrt{y - x^2} \) and sketch the domain on the given axes.

\[ y - x^2 \geq 0 \text{ so domain is all pts. (x, y) on or above the graph of } y = x^2 \]

b.) (5 pts.) Determine the range of \( f(x, y) = 9 - x^2 - y^2 \).

\[ z = 9 - (x^2 + y^2) \text{ and } 0 \leq x^2 + y^2 < \infty \]

so range is all \( z \leq 9 \)

9.) (10 pts.) You are to construct an open (no top) rectangular box of volume 4 ft.\(^3\). What should be the length \( x \), width \( y \), and height \( z \) of the box if the box is to have a minimum surface area? What is the minimum surface area? (You need only determine the critical point and surface area. You need NOT verify that it determines a minimum value.)

\[
\begin{align*}
\text{Volume } & \quad xyz = 4 \rightarrow \\
\text{ } & \quad z = \frac{4}{xy} \quad \text{minimize surface area} \\
S & = xy + 2xz + 2yz \\
& = xy + (2x + 2y)z \\
& = xy + (2x + 2y) \frac{4}{xy} \\
& = xy + \frac{8x}{xy} + \frac{8y}{xy} \\
& \rightarrow S = xy + \frac{8}{y} + \frac{8}{x} \\
\end{align*}
\]

\[
\begin{cases}
S_x = y - \frac{8}{x^2} = 0 \rightarrow y = \frac{8}{x^2} \\
S_y = x - \frac{8}{y^2} = 0 \rightarrow x = \frac{8}{y^2} \\
\end{cases}
\]

\[
x = \frac{8}{\left( \frac{8}{x^2} \right)^2} = \frac{8}{64x^4} = \frac{1}{8} x^4 \rightarrow 8x = x^4 \rightarrow x^4 - 8x = 0 \rightarrow x(x^3 - 8) = 0 \rightarrow x = 0 \text{ (no)} \text{ or } x = 2 \text{ ft.} \]

\[
y = 2 \text{ ft.} \quad z = 1 \text{ ft.} \text{ and minimum } S = 12 \text{ ft.}^2.
\]
The following EXTRA CREDIT PROBLEM is worth 8 points. This problem is OPTIONAL.

1.) Find a function $z = f(x, y)$ for which 

$$z_x = x^2y^2 \cos(xy) + 2xy \sin(xy) + 6x \quad \text{and} \quad z_y = x^3y \cos(xy) + x^2 \sin(xy) - 4,$$

or explain why it is not possible.

$$
z_x = x^2y \cdot \frac{\partial}{\partial x}[\sin(xy)] + 2xy \cdot \frac{\partial}{\partial x}[\sin(xy)] + 6x \\
= x^2y \cdot \frac{\partial}{\partial x}[\sin(xy)] + 2xy \cdot \frac{\partial}{\partial x}\left(x^2y\right) \sin(xy) + 6x \\
= \frac{\partial}{\partial x}\left(x^2y \cdot \sin(xy)\right) + 6x
$$

$$\Rightarrow z = x^2y \sin(xy) + 3x^2 + g(y)$$

$$\Rightarrow z_y = x^2y \cdot \cos(xy) \cdot x + x^2 \sin(xy) \cdot 0 + g'(y) \\
= x^3y \cos(xy) + x^2 \sin(xy) - 4$$

so $g'(y) = -4 \Rightarrow g(y) = -4y$.

Then

$$z = x^2y \sin(xy) + 3x^2 - 4y \quad \text{works}.$$