A Basic List of Power Series

Example 6 illustrates an important point in determining power series representations of functions. Although Taylor's Theorem is applicable to a wide variety of functions, it is often tedious to use because of the complexity of finding derivatives. The most practical use of Taylor's Theorem is in developing power series for a *basic list* of elementary functions. Then, from the basic list, you can determine power series for other functions by the operations of addition, subtraction, multiplication, division, differentiation, integration, and composition with known power series.

Power Series for Elementary Functions

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots, \qquad 0 < x < 2$$

$$\frac{1}{x+1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots, \qquad -1 < x < 1$$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots, \qquad 0 < x \le 2$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots, \qquad -\infty < x < \infty$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots, \qquad -1 < x < 1$$

The last series in the list above is called a **binomial series.** Example 7 illustrates the use of such a series.

EXAMPLE 7 Using the Basic List of Power Series

Find the power series for

$$g(x) = \sqrt[3]{1+x},$$

centered at zero.

SOLUTION Using the binomial series

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$

with $k = \frac{1}{3}$, you can write

$$(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8x^4}{3^4 4!} + \cdots$$

which converges for $-1 < x < 1$.

TRY IT 7

Use the basic list of power series to find the power series for $g(x) = \sqrt{1+x}$, centered at zero.