Math 16C
Kouba
Exam 3

PRACTICE EXAM

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. COPYING ANSWERS FROM A CLASSMATE's EXAM IS A VIOLATION OF THE UNIVERSITY HONOR CODE. HAVING ANOTHER PERSON TAKE YOUR EXM FOR YOU IS A VIOLATION OF THE UNIVERSITY HONOR CODE. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or handouts may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. You will be graded on proper use of derivative and summation notation.

6. Put units on answers where units are appropriate.

7. Make sure that you have 7 pages, including the cover page.

8. You have until 12 p.m. sharp to finish the exam.

1.) (6 pts. each) Determine the nth term (starting with $n = 1$) of each of the following sequences.

a.) $3, 5, 7, 9, 11, \cdots$

$1\ 2\ 3\ 4\ 5$

$3 + 2(n-1) = \boxed{2n+1}$

b.) $(2/3)^4, (2/3)^5, (2/3)^6, (2/3)^7, (2/3)^8, \cdots$

$1\ \ 2\ \ 3\ \ 4\ \ 5$

$\boxed{\left(\frac{2}{3}\right)^{n+3}}$

$2^1\ 2^2\ 2^3\ 2^4\ 2^5$

c.) $\frac{2}{1}, \frac{-4}{4}, \frac{8}{9}, \frac{-16}{16}, \frac{32}{25}, \cdots$

$1^2\ 2^2\ 3^2\ 4^2\ 5^2$

$1\ 2\ 3\ 4\ 5$

$(-1)^{n+1} \frac{2^n}{n^2}$

2.) (7 pts. each) Determine whether each series converges or diverges. Briefly explain and name the test that you are using.

a.) $\sum_{n=1}^{\infty} n^{-4} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ so converges by p-series test since $p = 4 > 1$

b.) $\sum_{n=0}^{\infty} \frac{3^n}{(n+1)!}$

$$\lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+2)!}}{\frac{3^n}{(n+1)!}} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{3^n}$$

$$= \lim_{n\to\infty} \frac{3}{n+2} = \frac{3}{\infty} = 0 < 1$$ so converges by the ratio test

c.) $\sum_{n=1}^{\infty} \cos\left(\pi + \frac{1}{n^2}\right)$ $\lim_{n\to\infty} \cos\left(\pi + \frac{1}{n^2}\right)$

$= \cos(\pi + 0) = \cos\pi = -1 \neq 0$ so diverges by nth-term test

d.) $\sum_{n=2}^{\infty} \left(\frac{4}{3}\right)^n$ $r = \frac{4}{3} > 1$ so diverges by geometric series test

3.) (10 pts.) Use Taylor's Theorem to find the first four nonzero terms of the Taylor series centered at $c = 1$ for the function $f(x) = \ln x$.

$f'(x) = \frac{1}{x} = x^{-1}$, $a_n = \frac{f^{(n)}(1)}{n!} \rightarrow$

$f''(x) = -x^{-2}$,

$f'''(x) = 2x^{-3}$, $a_0 = \frac{f(1)}{0!} = \frac{\ln 1}{1} = \frac{0}{1} = 0$,

$f^{(4)}(x) = -6x^{-4}$; $a_1 = \frac{f'(1)}{1!} = \frac{1}{1} = 1$,

$a_2 = \frac{f''(1)}{2!} = \frac{-1}{2}$, $a_3 = \frac{f'''(1)}{3!} = \frac{2}{6} = \frac{1}{3}$,

$a_4 = \frac{f^{(4)}(1)}{4!} = \frac{-6}{24} = \frac{-1}{4}$ so

$\ln x = 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$

4.) (7 pts. each) Use shortcuts to find the first four nonzero terms of the Taylor series centered at $c = 0$ for each function.

a.) $f(x) = \dfrac{x^3}{1-x^2}$;

$$\frac{x^3}{1-x^2} = x^3 \cdot \frac{1}{1-(x^2)} = x^3\left(1+(x^2)+(x^2)^2+(x^2)^3+\cdots\right)$$

$$= x^3(1+x^2+x^4+x^6+\cdots)$$

$$= x^3+x^5+x^7+x^9+\cdots$$

b.) $f(x) = e^{-x}\sin(2x)$

$$e^{-x} \cdot \sin(2x) = \left(1+(-x)+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\cdots\right) \cdot \left((2x)-\frac{(2x)^3}{3!}+\cdots\right)$$

$$=\left(1-x+\tfrac{1}{2}x^2-\tfrac{1}{6}x^3+\cdots\right)\left(2x-\tfrac{4}{3}x^3+\cdots\right)$$

$$= 2x-2x^2+x^3-\tfrac{1}{3}x^4+\cdots$$

$$-\tfrac{4}{3}x^3+\tfrac{4}{3}x^4-\cdots$$

$$= 2x-2x^2-\tfrac{1}{3}x^3+x^4+\cdots$$

5.) (7 pts.) The following p-series converges. How many terms should be used to estimate the exact value of the series with error at most 0.0001 ?

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots$$

p-series error $\dfrac{n^{1-p}}{p-1}$ so

and $p = 3$ so

$$\frac{n^{1-3}}{3-1} \le 0.0001 \longrightarrow \frac{n^{-2}}{2} \le 0.0001 \longrightarrow n^2 \ge \frac{1}{0.0002}$$

$$\longrightarrow n \ge \sqrt{\frac{1}{0.0002}} \approx 70.7 \text{ so choose } \boxed{n=71}$$

6.) (7 pts.) Find the interval of convergence and radius of convergence for the following power series :

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n \cdot 4^n}$$

$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n \cdot 4^n}$ $\lim_{n\to\infty} \frac{|x+2|^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n \cdot 4^n}{|x+2|^n} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1}{4} |x+2|^1$

$= 1 \cdot \frac{1}{4} |x+2|^1 < 1 \longrightarrow |x+2| < 4 \longrightarrow -4 < x+2 < 4 \longrightarrow$

$\boxed{-6 < x < 2}$ and radius is $\boxed{R = 4}$

7.) (8 pts.) Determine the exact value of the following convergent series.

$$-\frac{4}{3} + 1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \cdots$$

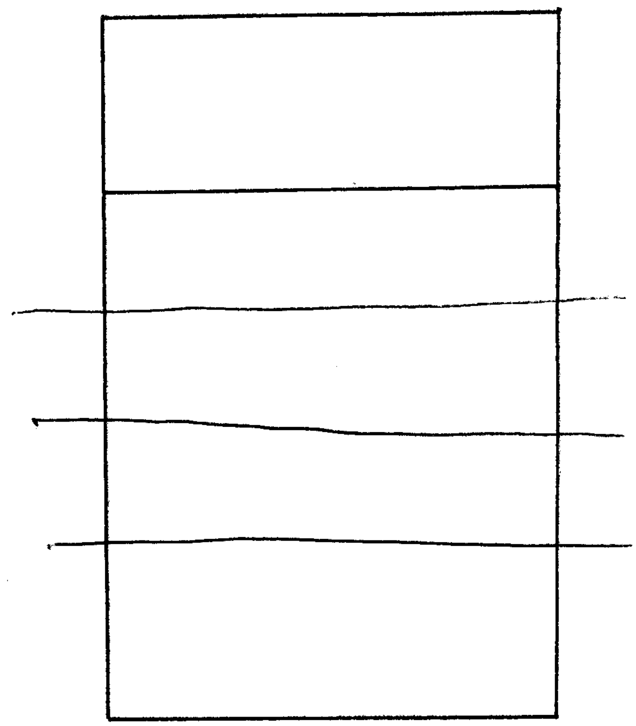
$= -\frac{4}{3}\left(1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \cdots\right)$

$= -\frac{4}{3}\left(1 + \left(-\frac{3}{4}\right) + \left(-\frac{3}{4}\right)^2 + \left(-\frac{3}{4}\right)^3 + \cdots\right)$

$= -\frac{4}{3} \cdot \frac{1}{1-\left(-\frac{3}{4}\right)} = -\frac{4}{3} \cdot \frac{1}{\frac{7}{4}}$

$= -\frac{4}{3} \cdot \frac{4}{7} = -\frac{16}{21}$

8.) (8 pts.) What is the maximum number of rectangles which can be formed within the boundary of the given figure using 200 horizontal lines ? Count all rectangles including overlapping ones.



# lines	# □'s
0	$1+2=3$
1	$1+2+3=6$
2	$1+2+3+4=10$
3	$1+2+3+4+5=15$
⋮	⋮
n	$1+2+3+\cdots+(n+2)$

$$= \frac{1}{2}(n+2)((n+2)+1)$$

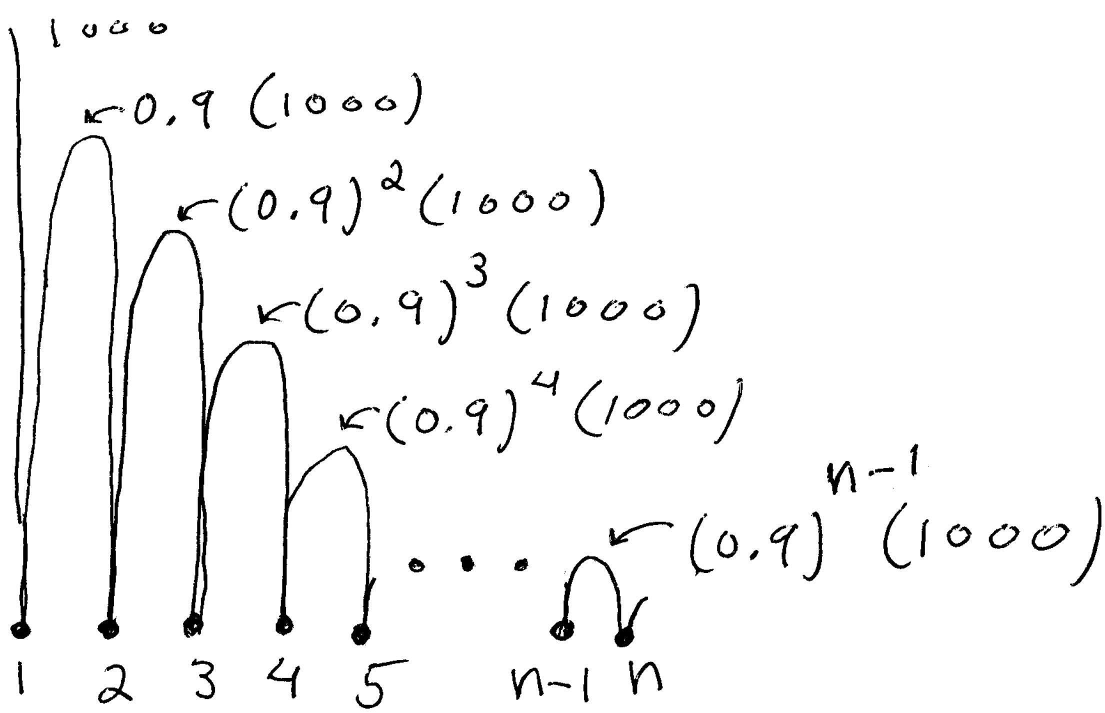
$$= \frac{1}{2}(n+2)(n+3) \; ;$$

if $n = 200$, then # of □'s is

$$\frac{1}{2}(200+2)(200+3) = 20{,}503$$

The following EXTRA CREDIT PROBLEM is worth 8 points. These problems are OPTIONAL.

1.) A rubber super ball has the property that if it is dropped from a height of H feet it will bounce $0.9H$ feet high. This super ball is dropped from a height of 1000 feet and bounces repeatedly. If the total distance that the ball travels vertically is 16,812 feet, determine the number of times n that the ball bounces.



$$1000 + 2(0.9)(1000) + 2(0.9)^2(1000) + 2(0.9)^3(1000) + \cdots + 2(0.9)^{n-1}(1000)$$

$$= 1000 + 2(0.9)(1000)\left[1 + (0.9) + (0.9)^2 + \cdots + (0.9)^{n-2}\right]$$

$$= 1000 + 1800 \cdot \frac{1-(0.9)^{n-1}}{1-0.9}$$

$$= 1000 + 18{,}000\,(1-0.9)^{n-1} = 16{,}812 \rightarrow$$

$$18{,}000\,(1-(0.9)^{n-1}) = 15{,}812 \rightarrow \quad 1-(0.9)^{n-1} = \frac{15{,}812}{18{,}000}$$

The following EXTRA CREDIT PROBLEM is worth 8 points. These problems are OPTIONAL.

1.) A rubber super ball has the property that if it is dropped from a height of H feet it will bounce $0.9H$ feet high. This super ball is dropped from a height of 1000 feet and bounces repeatedly. If the total distance that the ball travels vertically is 16,812 feet, determine the number of times n that the ball bounces.

$$\rightarrow \quad (0.9)^{n-1} = 1 - \frac{15{,}812}{18{,}000}$$

$$\rightarrow \quad \ln (0.9)^{n-1} = \ln \left(1 - \frac{15{,}812}{18{,}000}\right)$$

$$\rightarrow \quad (n-1) \ln (0.9) = \ln \left(\frac{2188}{18{,}000}\right)$$

$$\rightarrow \quad n-1 = \ln \left(\frac{2188}{18{,}000}\right) / \ln (0.9)$$

$$\rightarrow \quad n = 1 + \frac{\ln \left(\frac{2188}{18{,}000}\right)}{\ln (0.9)} \approx 21.001$$

$$\rightarrow \quad \boxed{21 \text{ bounces}}$$