Math 16C
Spring 1993
Final Exam

Name: KEY
Signature: ____________
Student Number: ____________

<u>Your 16C Instructor:</u> (check one)

- ☐ Borzellino
- ☐ Cohen
- ☐ Kouba
- ☐ Lorica
- ☐ Marx

<u>No Use of Texts, Notes, or Calculators Is Allowed</u>.
Show <u>all</u> work on these pages. Make sure that you have all <u>11</u> pages of problems (not counting this one).

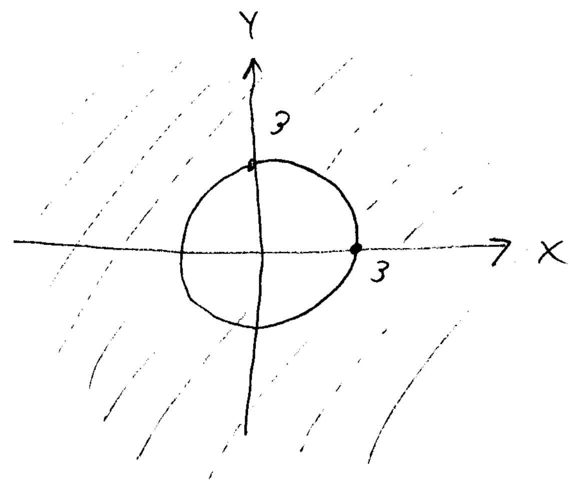
	Problem	Possible Points	Score
Cohen E	E 1	7	
	E 2 ab	10	
Borzzelino	* 3 ab	15	
Gu *	4	10	
	E 5 ab	12	
Lorica	M 6 abc	18	
Alexandrov	* 7	12	
	8	10	
Truong	* 9 ab	26	
Contos	* 10 ab	26	
Marx !	* 11	14	
Dance	M 12	13	
Winckler	M 13	13	
Kouba !	* 14	14	
	TOTAL	**200**	

1. [7 points] Find and sketch the domain of $f(x, y) = \sqrt{x^2 + y^2 - 9}$.

$x^2 + y^2 - 9 \geq 0$

$x^2 + y^2 \geq 3^2$

so domain is all pts. (x, y) on or outside the circle $x^2 + y^2 = 3^2$



2. [10 points] Let $f(x, y) = \cos(1 + xy^2)$. Compute the following:

(a) [4 points] f_y

$$f_y = -\sin(1 + xy^2) \cdot 2xy$$

$$= (-2xy) \cdot \sin(1 + xy^2)$$

(b) [6 points] f_{yx} .

$$f_{yx} = (-2xy) \cdot \cos(1 + xy^2) \cdot y^2$$

$$+ (-2y) \cdot \sin(1 + xy^2)$$

3. [15 points] Let $f(x, y) = 5 + x^2 - x^2y - y^2 - \frac{1}{3}y^3$.

(a) [8 points] Find all critical points f.

$f_x = 2x - 2xy = +2x(1-y) = 0 \rightarrow x=0$ or $y=1$

$f_y = -x^2 - 2y - y^2 = 0 \rightarrow x^2+y^2+2y=0$;

if $x=0$, then $y^2+2y=0 \rightarrow y(y+2)=0 \rightarrow$

$\boxed{(0,0)}$ and $\boxed{(0,-2)}$ are critical pts. ;

if $y=1$, then $x^2+3=0$ so no critical pts.

(b) [7 points] Classify each critical point of f as a relative maximum, relative minimum, or saddle point .

$f_{xx} = 2-2y$, $f_{yy} = -2-2y$, $f_{xy} = -2x$ so

$(0,0)$: $d = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (2)(-2) - (0)^2 = -4 < 0$ so

$(0,0)$ det. a saddle point at $z=5$;

$(0,-2)$: $d = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (6)(2) - (0)^2 = 12 > 0$ and

$f_{xx} = 6 > 0$ so $(0,-2)$ det. a minimum value

at $z = \frac{11}{3}$

4. [10 points] Use Newton's method with initial guess $x_1 = 0$ to compute two successive approximations to the solution of the equation $x^3 + 3x = 1$.

$f(x) = x^3 + 3x - 1 = 0 \rightarrow f'(x) = 3x^2 + 3$ then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 3x_n - 1}{3x_n^2 + 3}$$

$$= \frac{3x_n^3 + \cancel{3x_n} - x_n^3 - \cancel{3x_n} + 1}{3x_n^2 + 3} = \frac{2x_n^3 + 1}{3x_n^2 + 3} \; ; \; x_1 = 0 \text{ so}$$

$x_2 = \left(\frac{1}{3}\right)$ so

$$x_3 = \frac{2\left(\frac{1}{3}\right)^3 + 1}{3\left(\frac{1}{3}\right)^2 + 3} = \frac{\frac{2}{27} + 1}{\frac{1}{3} + 3} = \frac{\frac{29}{27}}{\frac{10}{3}} = \frac{29}{27} \cdot \frac{3}{10} = \frac{29}{90}$$

5. [12 points] Determine whether the following sequences converge or diverge. Find the limit of the convergent ones.

(a) $a_n = \frac{\sqrt{n^2 - 5}}{2n + 3}$

$$\lim_{n \to \infty} \sqrt{\frac{n^2 - 5}{4n^2 + 12n + 9}} = \lim_{n \to \infty} \sqrt{\frac{1 - 5/n^2}{4 + \frac{12}{n} + \frac{9}{n^2}}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

so converges

(b) $a_n = (-1)^n \cos(\frac{1}{n})$ $\quad \lim_{n \to \infty} \cos\left(\frac{1}{n}\right) = \cos 0 = 1$ so

$\lim_{n \to \infty} (-1)^n \cos\left(\frac{1}{n}\right)$ does not exist (diverges)

6. [18 points] Determine whether the following series converge or diverge. Clearly explain why.

(a) $\sum_{n=0}^{\infty} \frac{n}{500n + 79}$ $\lim_{n\to\infty} \frac{n}{500n+79} = \lim_{n\to\infty} \frac{1}{500 + \frac{79}{n}} = \frac{1}{500} \neq 0$

so series diverges by nth term test

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ so series converges

by p-series test with $p = 5/2 > 1$

(c) $\sum_{n=1}^{\infty} (2n)!\,(\frac{2}{3})^n$

ratio test: $\lim_{n\to\infty} \frac{(2(n+1))!\,\left(\frac{2}{3}\right)^{n+1}}{(2n)!\,\left(\frac{2}{3}\right)^{n}}$

$= \lim_{n\to\infty} (2n+2)(2n+1)\left(\frac{2}{3}\right) = +\infty > 1$ so

series diverges.

7. [12 points] Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{3^{-n}}{n+1}(x+1)^n$. (Do not check end points.)

$$\lim_{n\to\infty} \frac{3^{-(n+1)} \cdot |x+1|^{n+1}}{(n+2)} \cdot \frac{n+1}{3^{-n}|x+1|^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{3^{-n}}3^{-1}}{\cancel{3^{-n}}} \cdot \frac{1+\frac{1}{n}}{1+\frac{2}{n}} \cdot |x+1| = \left(\frac{1}{3}\right)\cdot\left(1\right)\cdot|x+1| = \frac{|x+1|}{3} < 1$$

$$\rightarrow |x+1| < 3 \rightarrow -3 < x+1 < 3 \rightarrow -4 < x < 2$$

and $R = 3$

8. [10 points] Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{3^n 2^{n-2}}$.

$$= \frac{1}{3^1 2^{-1}} + \frac{1}{3^2 \cdot 2^0} + \frac{1}{3^3 \cdot 2^1} + \frac{1}{3^4 \cdot 2^2} + \frac{1}{3^5 \cdot 2^3} + \cdots$$

$$= \frac{2}{3} + \frac{1}{3^2} \cdot \left[1 + \frac{1}{3^1 2^1} + \frac{1}{3^2 2^2} + \frac{1}{3^3 \cdot 2^3} + \cdots \right]$$

$$= \frac{2}{3} + \frac{1}{9} \cdot \left[1 + \left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^3 + \cdots \right]$$

$$= \frac{2}{3} + \frac{1}{9} \cdot \frac{1}{1-\frac{1}{6}} = \frac{2}{3} + \frac{1}{9} \cdot \frac{6}{5} = \frac{2}{3} + \frac{2}{15} = \frac{12}{15} = \frac{4}{5}$$

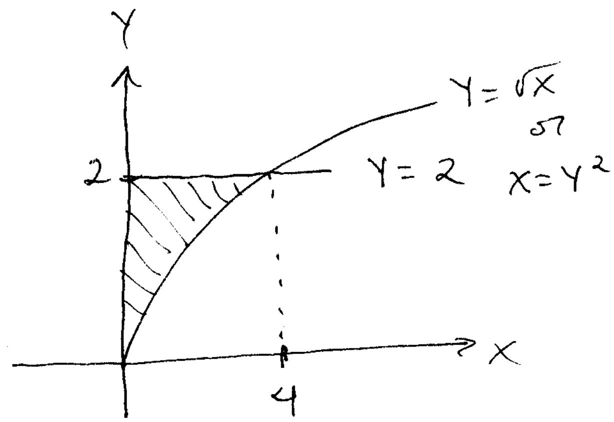
9. [26 points] Evaluate the following double integrals.

(a) $\int_0^2 \int_0^{\sqrt{x}} y(x-y^2)^3 \, dy \, dx$ $= \int_0^2 \frac{-1}{2} \cdot \frac{(x-y^2)^4}{4} \Big|_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^2 \left(\frac{-1}{8}(0) - \frac{-1}{8} x^4 \right) dx = \frac{1}{8} \int_0^2 x^4 \, dx$

$= \frac{1}{8} \cdot \frac{x^5}{5} \Big|_0^2 = \frac{32}{8 \cdot 5} = \frac{4}{5}$

(b) $\int_0^4 \int_{\sqrt{x}}^2 x \sin(1+y^5) \, dy \, dx$



$= \int_0^2 \int_0^{y^2} x \sin(1+y^5) \, dx \, dy$

$= \int_0^2 \frac{x^2}{2} \sin(1+y^5) \Big|_{x=0}^{x=y^2} dy$

$= \frac{1}{2} \int_0^2 y^4 \sin(1+y^5) \, dx$

$= \frac{1}{2} \cdot \frac{-1}{5} \cos(1+y^5) \Big|_0^2$

$= \frac{-1}{10} (\cos 33 - \cos 1)$

10. [26 points] Solve the following differential equations.

(a) $e^{x^2-x}y' + y = 2xy \longrightarrow e^{x^2-x}Y' = 2XY - Y \longrightarrow$

$$e^{x^2-x}Y' = Y(2X-1) \rightarrow \int \frac{1}{Y}\,dy = \int \frac{2X-1}{e^{X^2-X}}\,dx \rightarrow$$

$$\ln|Y| = -e^{X-X^2} + c$$

or

$$Y = c\,e^{-e^{X-X^2}}$$

(b) $xy' - 2y = x \ln x, \quad y(1) = 0$

$$Y' + \left(\frac{-2}{X}\right)Y = \ln X \rightarrow \mu = e^{\int \frac{-2}{X}dx} = e^{-2\ln X} = X^{-2} \rightarrow$$

$$X^{-2}Y' - 2X^{-3}Y = X^{-2}\ln X \longrightarrow$$

$$D\left(\frac{1}{X^2}Y\right) = \frac{\ln X}{X^2} \rightarrow \frac{1}{X^2}Y = \int \frac{\ln X}{X^2}\,dx \longrightarrow$$

(Let $u = \ln X$, $dV = \frac{1}{X^2}\,dx$

$du = \frac{1}{X}\,dx$, $V = \frac{-1}{X}$)

$$\frac{1}{X^2}Y = \frac{-\ln X}{X} + \int \frac{1}{X^2}\,dx \rightarrow \frac{1}{X^2}Y = \frac{-\ln X}{X} - \frac{1}{X} + c \rightarrow$$

$$Y = -X\ln X - X + cX^2, \quad Y(1) = 0 \longrightarrow$$

$$0 = -\cancel{\ln 1}^{0} - 1 + c \rightarrow c = 1 \longrightarrow$$

$$\boxed{Y = -X\ln X - X + X^2}$$

11. [~~12~~ 14 points] John is supposed to learn 1,000 French vocabulary words, of which he initially knows none. Suppose that he learns these words at a rate proportional to the number of words that he has not yet learned, and that he learns 150 words in the first 5 days. How many days does it take him to learn half the words? (Let N be the number of words learned after t days.)

$$\frac{dN}{dt} = k(1000 - N), \qquad N(0) = 0 \text{ words}, \quad N(5) = 150 \text{ words}$$

$$\int \frac{1}{1000-N}\, dN = \int k\, dt \rightarrow -\ln|1000-N| = kt + c \rightarrow$$

$$\ln|1000-N| = -kt + c_1 \rightarrow$$

$$1000 - N = e^{-kt+c_1} = c_2 e^{-kt} \rightarrow$$

$$N = 1000 + Ce^{-kt}; \quad N(0) = 0 \rightarrow C = -1000 \text{ so}$$

$$N = 1000 - 1000e^{-kt}, \quad N(5) = 150 \rightarrow$$

$$150 = 1000 - 1000e^{-5k} \rightarrow 1000e^{-5k} = 850 \rightarrow$$

$$e^{-5k} = 0.85 \rightarrow -5k = \ln(0.85) \rightarrow k = \frac{\ln(0.85)}{-5}$$

so $\boxed{N = 1000 - 1000e^{\frac{\ln(0.85)}{5}t}}$; if $N = 500$ then

$$500 = 1000 - 1000e^{\frac{\ln(0.85)}{5}t} \rightarrow \frac{1}{2} = e^{\frac{\ln(0.85)}{5}t} \rightarrow$$

$$\ln\left(\tfrac{1}{2}\right) = \frac{\ln(0.85)}{5}t \rightarrow \boxed{t = \frac{5\ln(1/2)}{\ln(0.85)} \text{ days}}$$

12. [13 points] Use any method to find the 3rd–degree Taylor polynomial centered at $c = 1$ for $f(x) = \frac{1}{2x-1} = (2x-1)^{-1}$, $f'(x) = -2(2x-1)^{-2}$,

$f''(x) = 8(2x-1)^{-3}$, $f'''(x) = -48(2x-1)^{-4}$, ..., $a_n = \frac{f^{(n)}(1)}{n!} \rightarrow$

$a_0 = f(1) = 1$, $a_1 = f'(1) = -2$, $a_2 = \frac{f''(1)}{2!} = \frac{8}{2} = 4$,

$a_3 = \frac{f'''(1)}{3!} = \frac{-48}{6} = -8$ so

$$\boxed{P_3(x) = 1 - 2(x-1) + 4(x-1)^2 - 8(x-1)^3}$$

13. [13 points] Approximate the definite integral $\int_0^1 xe^{-x^3}\,dx$ using a 7th–degree Taylor polynomial for $f(x) = xe^{-x^3}$. Express your answer as a fraction.

$e^x = 1 + x + \frac{x^2}{2!} + \cdots$, $e^{-x^3} = 1 - x^3 + \frac{x^6}{2!} - \cdots$,

$xe^{-x^3} = x - x^4 + \frac{x^7}{2!} - \cdots$ so

$$\int_0^1 xe^{-x^3}dx \approx \int_0^1 \left(x - x^4 + \frac{x^7}{2}\right)dx = \left(\frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{16}\right)\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{5} + \frac{1}{16} = \frac{40 - 16 + 5}{80} = \frac{29}{80}$$

14. [~~12~~ 14 points] Use the method of Lagrange multipliers to minimize

$f(x, y, z) = x^2 + y^2 + z^2$ subject to $x - y + 2z = 3$ and $3x + y - z = 0$.

$$F(x,y,z,\mu,\lambda) = x^2+y^2+z^2-\mu(x-y+2z-3)-\lambda(3x+y-z)$$

$$F_x = 2x-\mu-3\lambda=0$$
$$F_y = 2y+\mu-\lambda=0$$
$$F_z = 2z-2\mu+\lambda=0$$
$$F_\mu = -x+y-2z+3=0$$
$$F_\lambda = -3x-y+z=0$$

$$2x+2y-4\lambda=0 \rightarrow \lambda = \tfrac{1}{2}x+\tfrac{1}{2}y$$
$$4y+2z-\lambda=0 \rightarrow \lambda = 4y+2z$$

$$\tfrac{1}{2}x+\tfrac{1}{2}y = 4y+2z \rightarrow$$
$$x+y = 8y+4z \rightarrow$$

$$\left.\begin{aligned} x-7y-4z&=0\\ -x+y-2z&=-3\\ -3x-y+z&=0 \end{aligned}\right\}$$

$$\left.\begin{aligned} -6y-6z&=-3\\ -4y+7z&=9 \end{aligned}\right\} \quad \left.\begin{aligned} 2y+2z&=1\\ -4y+7z&=9 \end{aligned}\right\}$$

$$11z=11 \rightarrow z=1,\ y=\tfrac{-1}{2},\ x=\tfrac{1}{2}$$

and min. : $f(1/2, \tfrac{-1}{2}, 1) = \tfrac{1}{4}+\tfrac{1}{4}+1 = 3/2$

Appendix —— Some formulas you might use

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad \text{for } -1 < x < 1 ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \qquad \text{for } -1 < x < 1 ,$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \text{for all } x ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \qquad \text{for all } x ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \qquad \text{for all } x ,$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots \qquad \text{for } -1 < x < 1 .$$