The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, evaluate the function when \( x = -3 \).

1. \( f(x) = 5 - 2x \)
2. \( f(x) = -x^2 + 4x + 5 \)
3. \( y = \sqrt{4x^2 - 3x + 4} \)
4. \( y = \sqrt[3]{34 - 4x + 2x^2} \)

In Exercises 5–8, find the domain of the function.

5. \( f(x) = 5x^2 + 3x - 2 \)
6. \( g(x) = \frac{1}{2x} - \frac{2}{x + 3} \)
7. \( h(y) = \sqrt{y - 5} \)
8. \( f(y) = \sqrt{y^2 - 5} \)

In Exercises 9 and 10, evaluate the expression.

9. \((476)^{0.55}\)
10. \((25)^{0.55}\)

In Exercises 15–18, determine the plane that corresponds to the range of the function.

15. \( f(x, y) = \sqrt{16 - x} \)
16. \( f(x, y) = x^2 + y^2 \)
17. \( f(x, y) = e^{xy} \)
18. \( f(x, y) = \ln(x + y) \)

In Exercises 19–28, determine the plane that corresponds to the range of the function.

19. \( f(x, y) = \sqrt[3]{9 - 9x} \)
20. \( f(x, y) = \sqrt{x^2 + y} \)
21. \( f(x, y) = \frac{x}{y} \)
22. \( f(x, y) = \frac{4y}{x - 1} \)
23. \( f(x, y) = \frac{1}{xy} \)
24. \( g(x, y) = \frac{1}{x - y} \)
25. \( h(x, y) = x\sqrt{y} \)
26. \( f(x, y) = \sqrt{xy} \)
27. \( g(x, y) = \ln(4 - x) \)
28. \( f(x, y) = ye^{\ln x} \)

In Exercises 29–32, mat the contour maps. (The contour maps are given in the text.)

29. \( f(x, y) = x^2 + \frac{y^2}{4} \)
30. \( f(x, y) = \sqrt{x^2 + y^2} \)
31. \( f(x, y) = e^{1-x^2-y^2} \)
In Exercises 15–18, describe the region $R$ in the $xy$-coordinate plane that corresponds to the domain of the function, and find the range of the function.

15. $f(x, y) = \sqrt{16 - x^2 - y^2}$
16. $f(x, y) = x^2 + y^2 - 1$
17. $f(x, y) = e^{xy}$
18. $f(x, y) = \ln(x + y)$

In Exercises 19–28, describe the region $R$ in the $xy$-coordinate plane that corresponds to the domain of the function.

19. $f(x, y) = \sqrt{9 - 9x^2 - y^2}$
20. $f(x, y) = \sqrt{x^2 + y^2} - 1$
21. $f(x, y) = \frac{x}{y}$
22. $f(x, y) = \frac{4y}{x - 1}$
23. $f(x, y) = \frac{1}{xy}$
24. $g(x, y) = \frac{1}{x - y}$
25. $h(x, y) = x\sqrt{y}$
26. $f(x, y) = \sqrt{xy}$
27. $g(x, y) = \ln(4 - x - y)$
28. $f(x, y) = ye^{1/x}$

In Exercises 29–32, match the graph of the surface with one of the contour maps. [The contour maps are labeled (a)–(d).]

29. $f(x, y) = x^2 + \frac{y^2}{4}$
30. $f(x, y) = e^{1-x^2+y^2}$

31. $f(x, y) = e^{1-x^2-y^2}$
32. $f(x, y) = \ln|y - x^2|$

In Exercises 33–40, describe the level curves of the function. Sketch the level curves for the given $c$-values.

- **Function**
  - $z = x + y$
  - $z = 6 - 2x - 3y$
  - $z = \sqrt{16 - x^2 - y^2}$
  - $f(x, y) = x^2 + y^2$
  - $f(x, y) = xy$
  - $z = e^{xy}$
  - $f(x, y) = \frac{x}{x^2 + y^2}$
  - $f(x, y) = \ln(x - y)$

- **$c$-Values**
  - $c = -1, 0, 2, 4$
  - $c = 0, 2, 4, 6, 8, 10$
  - $c = 0, 1, 2, 3, 4$
  - $c = 0, 2, 4, 6, 8$
  - $c = \pm 1, \pm 2, \ldots, \pm 6$
  - $c = 1, 2, 3, 4, \frac{1}{2}, \frac{1}{4}$
  - $c = \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2$
  - $c = 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2$

41. **Cobb-Douglas Production Function** A manufacturer estimates the Cobb-Douglas production function to be given by

$$f(x, y) = 100x^{0.75}y^{0.25}.$$ Estimate the production levels when $x = 1500$ and $y = 1000$.

42. **Cobb-Douglas Production Function** Use the Cobb-Douglas production function (Example 5) to show that if both the number of units of labor and the number of units of capital are doubled, the production level is also doubled.

43. **Cost** A company manufactures two types of woodburning stoves: a freestanding model and a fireplace-insert model. The cost function for producing $x$ freestanding stoves and $y$ fireplace-insert stoves is given by

$$C(x, y) = 27\sqrt{xy} + 195x + 215y + 980.$$ Find the cost when $x = 80$ and $y = 20$. 
44. Forestry The Doyle Log Rule is one of several methods used to determine the lumber yield of a log in board-feet in terms of its diameter \( d \) in inches and its length \( L \) in feet. The number of board-feet is given by

\[ N(d, L) = \left( \frac{d - 4}{4} \right)^2 L. \]

(a) Find the number of board-feet of lumber in a log with a diameter of 22 inches and a length of 12 feet.

(b) Find \( N(30, 12) \).

45. Profit A sporting goods manufacturer produces regulation soccer balls at two plants. The costs of producing \( x_1 \) units at location 1 and \( x_2 \) units at location 2 are given by

\[ C_1(x_1) = 0.02x_1^2 + 4x_1 + 500 \]

and

\[ C_2(x_2) = 0.05x_2^2 + 4x_2 + 275 \]

respectively. If the product sells for $45 per unit, then the profit function for the product is given by

\[ P(x_1, x_2) = 45(x_1 + x_2) - C_1(x_1) - C_2(x_2). \]

Find (a) \( P(250, 150) \) and (b) \( P(300, 200) \).

46. Consumer Awareness The average amount of time that a customer waits in line for service is given by

\[ W(x, y) = \frac{1}{x - y}, \quad y < x \]

where \( y \) is the average arrival rate and \( x \) is the average service rate (\( x \) and \( y \) are measured in the number of customers per hour). Evaluate \( W \) at each point.

(a) (15, 10)  (b) (12, 9)  (c) (12, 6)  (d) (4, 2)

47. Investment In 2004, an investment of $1000 was made in a bond earning 10% compounded annually. The investor pays tax at rate \( R \), and the annual rate of inflation is \( I \). In the year 2014, the value \( V \) of the bond in constant 2004 dollars is given by

\[ V(I, R) = 1000 \left[ \frac{1 + 0.10(1 - R)}{1 + I} \right]^{10}. \]

Use this function of two variables to complete the table.

<table>
<thead>
<tr>
<th>Tax rate, ( R )</th>
<th>Inflation rate, ( I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.28</td>
<td></td>
</tr>
<tr>
<td>0.35</td>
<td></td>
</tr>
</tbody>
</table>

48. Meteorology Meteorologists measure the atmospheric pressure in millibars. From these observations they create weather maps on which the curves of equal atmospheric pressure (isobars) are drawn (see figure). On the map, the closer the isobars the higher the wind speed. Match points \( A, B, \) and \( C \) with (a) highest pressure, (b) lowest pressure, and (c) highest wind velocity.

49. Geology: A Contour Map The contour map below represents color-coded seismic amplitudes of a fault horizon and a projected contour map, which is used in earthquake studies. (Source: Adapted from Shipman/Wilson/Feld. An Introduction to Physical Science, Tenth Edition)

50. Partial Derivatives

If \( z = f(x, y) \), \( t \) and \( y \) are the variables, such as:

\[ \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{z(x + \Delta x, y) - z(x, y)}{\Delta x} \]

\[ \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{z(x, y + \Delta y) - z(x, y)}{\Delta y} \]

Evaluate the partial derivatives as it has inde

---

**Functions of Several Variables**

Real-life applications show how changes in certain variables, such as:

You can follow with respect to one function with respect to another. This process partial derivatives as it has inde

---

**Example 41**

Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \)

**Solution**

\[ \frac{\partial z}{\partial x} = 3 - 2xy \]

\[ \frac{\partial z}{\partial y} = -2x^2y \]

**Try It 1**

(a) Discuss the use of color to represent the level curves.

(b) Do the level curves correspond to equally spaced amplitudes? Explain your reasoning.

---

**Example 50**

The earnings per share for Starbucks Corporation from 1995 through 2003 can be modeled by

\[ z = 0.265x - 0.209y + 0.033 \]

where \( x \) is sales (in billions of dollars) and \( y \) is the shareholder's equity (in billions of dollars). (Source: Starbucks Corporation)

(a) Find the earnings per share when \( x = 8 \) and \( y = 5 \).

(b) Which of the two variables in this model has the greater influence on the earnings per share? Explain your reasoning.
In Exercises 1–8, find the derivative of the function.

1. \( f(x) = \sqrt{x^2 + 3} \)
2. \( g(x) = (3 - x^2)^3 \)
3. \( g(t) = te^{2t+1} \)
4. \( f(x) = e^{2x} \sqrt{1 - e^{2x}} \)
5. \( f(x) = \ln(3 - 2x) \)
6. \( u(t) = \ln\sqrt{t^3 - 6t} \)
7. \( g(x) = \frac{5x^2}{(4x - 1)^2} \)
8. \( f(x) = \frac{(x + 2)^3}{(x^2 - 9)^2} \)

In Exercises 9 and 10, evaluate the derivative at the point (2, 4).

9. \( f(x) = x^2e^{x-2} \)
10. \( g(x) = x\sqrt{x^2 - x} + 2 \)

In Exercises 1–14, find the first partial derivatives with respect to \( x \) and with respect to \( y \).

1. \( f(x, y) = 2x - 3y + 5 \)
2. \( f(x, y) = x^2 - 3y^2 + 7 \)
3. \( f(x, y) = 5\sqrt{x} - 6y^2 \)
4. \( f(x, y) = x^{-1/2} + 4y^{3/2} \)
5. \( f(x, y) = \frac{x}{y} \)
6. \( z = x\sqrt{y} \)
7. \( f(x, y) = \sqrt{x^2 + y^2} \)
8. \( f(x, y) = \frac{xy}{x^2 + y^2} \)
9. \( z = x^2e^{2y} \)
10. \( z = xe^{-xy} \)
11. \( h(x, y) = e^{-(x^2+y^2)} \)
12. \( g(x, y) = xe^{xy} \)
13. \( z = \ln\frac{x - y}{(x+y)^2} \)
14. \( g(x, y) = \ln\sqrt{x^2 + y^2} \)

In Exercises 15–20, let \( f(x, y) = 3x^2ye^{x-y} \) and \( g(x, y) = 3xy^2e^{x-y} \). Find each of the following.

15. \( f_x(x, y) \)
16. \( f_y(x, y) \)
17. \( g_x(x, y) \)
18. \( g_y(x, y) \)
19. \( f_x(1, 1) \)
20. \( g_y(-2, -2) \)

In Exercises 21–28, evaluate \( f_x \) and \( f_y \) at the point.

Function | Point
---|---
21. \( f(x, y) = 3x^2 + xy - y^2 \) | (2, 1)
22. \( f(x, y) = x^2 - 3xy + y^2 \) | (1, -1)
23. \( f(x, y) = e^{3xy} \) | (0, 4)
24. \( f(x, y) = e^{x^2y^2} \) | (0, 2)

In Exercises 29–32, find the first partial derivatives with respect to \( x, y, \) and \( z \).

29. \( w = 3x^2y - 5xyz + 10yz^2 \)
30. \( w = \sqrt{x^2 + y^2 + z^2} \)
31. \( w = \frac{xy}{x + y + z} \)
32. \( w = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}} \)

In Exercises 33–38, evaluate \( w_x, w_y, \) and \( w_z \) at the point.

Function | Point
---|---
33. \( w = \sqrt{x^2 + y^2 + z^2} \) | (2, -1, 2)
34. \( w = \frac{xy}{x + y + z} \) | (1, 2, 0)
35. \( w = \ln\sqrt{x^2 + y^2 + z^2} \) | (3, 0, 4)
36. \( w = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}} \) | (0, 0, 0)
37. \( w = 2x^2 + 3xy - 6y^2z \) | (1, -1, 2)
38. \( w = xye^{2z} \) | (2, 1, 0)
In Exercises 39–42, find values of \( x \) and \( y \) such that \( f_x(x, y) = 0 \) and \( f_y(x, y) = 0 \) simultaneously.

39. \( f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3 \)
40. \( f(x, y) = 3x^3 - 12xy + y^3 \)
41. \( f(x, y) = \frac{1}{x} + \frac{1}{y} + xy \)
42. \( f(x, y) = \ln(x^2 + y^2 + 1) \)

In Exercises 43–50, find the slope of the surface at the given point in (a) the \( x \)-direction and (b) the \( y \)-direction.

<table>
<thead>
<tr>
<th>Function</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>43. ( z = 2x - 3y + 5 )</td>
<td>(2, 1, 6)</td>
</tr>
<tr>
<td>44. ( z = xy )</td>
<td>(1, 2, 2)</td>
</tr>
<tr>
<td>45. ( z = x^2 - 9y^2 )</td>
<td>(3, 1, 0)</td>
</tr>
<tr>
<td>46. ( z = x^2 + 4y^2 )</td>
<td>(2, 1, 8)</td>
</tr>
<tr>
<td>47. ( z = \sqrt{25 - x^2 - y^2} )</td>
<td>(3, 0, 4)</td>
</tr>
<tr>
<td>48. ( z = \frac{x}{y} )</td>
<td>(3, 1, 3)</td>
</tr>
<tr>
<td>49. ( z = 4 - x^2 - y^2 )</td>
<td>(1, 1, 2)</td>
</tr>
</tbody>
</table>

50. \( z = x^2 - y^2 \) \hspace{1cm} (-2, 1, 3)

In Exercises 51–54, show that \( \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \).

51. \( z = x^2 - 2xy + 3y^2 \)
52. \( z = x^4 - 3x^2y^2 + y^4 \)

53. \( z = \frac{e^{2xy}}{4x} \)
54. \( z = \frac{x^2 - y^2}{2xy} \)

In Exercises 55–62, find the second partial derivatives

\[ \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial y \partial x} \]
and \( \frac{\partial^2 z}{\partial x \partial y} \).

55. \( z = x^2 - 4y^2 \)
56. \( z = 3x^3 - xy + 2y^3 \)
57. \( z = 4x^3 + 3xy^2 - 4y^3 \)
58. \( z = \sqrt{9 - x^2 - y^2} \)
59. \( z = \frac{xy}{x - y} \)
60. \( z = \frac{x}{x + y} \)
61. \( z = xe^{-y^2} \)
62. \( z = xe^x + ye^x \)

In Exercises 63–66, evaluate the second partial derivatives \( f_{xx}, f_{xy}, f_{yx}, f_{yy} \) at the point.

<table>
<thead>
<tr>
<th>Function</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>63. ( f(x, y) = x^4 - 3x^2y^2 + y^2 )</td>
<td>(1, 0)</td>
</tr>
<tr>
<td>64. ( f(x, y) = \sqrt{x^2 + y^2} )</td>
<td>(0, 2)</td>
</tr>
<tr>
<td>65. ( f(x, y) = \ln(x - y) )</td>
<td>(2, 1)</td>
</tr>
<tr>
<td>66. ( f(x, y) = x^2e^y )</td>
<td>(-1, 0)</td>
</tr>
</tbody>
</table>

67. Marginal Cost  A company manufactures two models of bicycles: a mountain bike and a racing bike. The cost function for producing \( x \) mountain bikes and \( y \) racing bikes is given by

\[ C = 10\sqrt{xy} + 149x + 189y + 675. \]

Find the marginal costs \( \frac{\partial C}{\partial x} \) and \( \frac{\partial C}{\partial y} \) when \( x = 120 \) and \( y = 160 \).

68. Marginal Revenue A pharmaceutical corporation has two plants that produce the same over-the-counter medicine. If \( x_1 \) and \( x_2 \) are the numbers of units produced at plant 1 and plant 2, respectively, then the total revenue for the product is given by

\[ R = 200x_1 + 200x_2 - 4x_1^2 - 8x_1x_2 - 4x_2^2. \]

If \( x_1 = 4 \) and \( x_2 = 12 \), find the following.
(a) The marginal revenue for plant 1, \( \frac{\partial R}{\partial x_1} \)
(b) The marginal revenue for plant 2, \( \frac{\partial R}{\partial x_2} \)

69. Marginal Proc the Cobb-Dougl
\( f(x, y) = 100x^b \)
(a) Find the mar
(b) Find the mar
70. Marginal Pro production funct
71. Complementar notation of Exar demands for pro the prices of pr whether the dem tary or substitute
(a) \( x_1 = 150 - \)
(b) \( x_1 = 150 - \)
(c) \( x_1 = \frac{1000}{\sqrt{P_1P_2}} \)
72. Psychology E gence test called known as the IQ vidual's mental chronological ag The result is the
\[ IQ(M, C) = \frac{A}{c} \]
Find the partial d respect to C. Ev (12, 10) and inte
73. Education Le versity, \( p \) the char and t the tuition. such that \( \frac{\partial N}{\partial p} \) interpret the fact
74. Chemistry Th plate is given by
\[ T = 500 - 0. \]
where \( x \) and \( y \) an find the rate of c the distance mov x- and y-axes.
75. Chemistry A t two average pers model for this inc
\[ A = 0.885t - \]
where \( A \) is the a ture, and \( h \) is ti
69. Marginal Productivity Let \( x = 1000 \) and \( y = 500 \) in the Cobb-Douglas production function given by
\[
f(x, y) = 100x^{0.6}y^{0.4}
\]
(a) Find the marginal productivity of labor, \( \frac{\partial f}{\partial x} \).
(b) Find the marginal productivity of capital, \( \frac{\partial f}{\partial y} \).

70. Marginal Productivity Repeat Exercise 69 for the production function given by \( f(x, y) = 100x^{0.75}y^{0.25} \).

71. Complementary and Substitute Products Using the notation of Example 4 in this section, let \( x_1 \) and \( x_2 \) be the demands for products 1 and 2, respectively, and \( p_1 \) and \( p_2 \) the prices of products 1 and 2, respectively. Determine whether the demand functions below describe complementary or substitute product relationships.
(a) \( x_1 = 150 - 2p_1 - \frac{5}{3}p_2 \), \( x_2 = 350 - \frac{3}{5}p_1 - 3p_2 \)
(b) \( x_1 = 150 - 2p_1 + 1.8p_2 \), \( x_2 = 350 + \frac{2}{3}p_1 - 1.9p_2 \)
(c) \( x_1 = \frac{1000}{\sqrt{p_1p_2}} \), \( x_2 = \frac{750}{p_2\sqrt{p_1}} \)

72. Psychology Early in the twentieth century, an intelligence test called the Stanford-Binet Test (more commonly known as the IQ test) was developed. In this test, an individual's mental age \( M \) is divided by the individual's chronological age \( C \) and the quotient is multiplied by 100. The result is the individual's IQ.
\[
IQ(M, C) = \frac{M}{C} \times 100
\]
Find the partial derivatives of IQ with respect to \( M \) and with respect to \( C \). Evaluate the partial derivatives at the point \((12, 10)\) and interpret the result. (Source: Adapted from Bernstein/Clark-Stewart/Ray/Wickens, Psychology, Fourth Edition)

73. Education Let \( N \) be the number of applicants to a university, \( p \) the charge for food and housing at the university, and \( t \) the tuition. Suppose that \( N \) is a function of \( p \) and \( t \) such that \( \frac{\partial N}{\partial p} < 0 \) and \( \frac{\partial N}{\partial t} < 0 \). How would you interpret the fact that both partials are negative?

74. Chemistry The temperature at any point \((x, y)\) in a steel plate is given by
\[
T = 500 - 0.6x^2 - 1.5y^2
\]
where \( x \) and \( y \) are measured in meters. At the point \((2, 3)\), find the rate of change of the temperature with respect to the distance moved along the plate in the directions of the \( x \) and \( y \)-axes.

75. Chemistry A measure of what hot weather feels like to two average persons is the Apparent Temperature Index. A model for this index is
\[
A = 0.885t - 78.7h + 1.200h + 2.70
\]
where \( A \) is the apparent temperature, \( t \) is the air temperature, and \( h \) is the relative humidity in decimal form. (Source: The UMAP Journal)

76. Marginal Utility The utility function \( U = f(x, y) \) is a measure of the utility (or satisfaction) derived by a person from the consumption of two goods \( x \) and \( y \). Suppose the utility function is given by
\[
U = -5x^2 + xy - 3y^2
\]
(a) Find \( \frac{\partial U}{\partial t} \) and \( \frac{\partial U}{\partial h} \) when \( t = 90^\circ F \) and \( h = 0.80 \).
(b) Which has a greater effect on \( U \), air temperature or humidity? Explain your reasoning.

77. Research Project Use your school’s library, the Internet, or some other reference source to research a company that increased the demand for its product by creative advertising. Write a paper about the company. Use graphs to show how a change in demand is related to a change in the marginal utility of a product or service.

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**BUSINESS CAPSULE**

Fred and Richard Calloway of Augusta, Georgia, cofounded Male Care, which provides barber, dry cleaning, and car wash services. Among the many advertising techniques used by the Calloways to attract new clients are coupons, customer referrals, and radio advertising. They also feel that their prime location provides them with a strong customer base. Eighty percent of their 1800 monthly clients are repeat customers.
### PREREQUISITE REVIEW 7.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–8, solve the system of equations.

1. \[
\begin{cases}
5x = 15 \\
3x - 2y = 5
\end{cases}
\]

2. \[
\begin{cases}
\frac{1}{2}y \neq 3 \\
x - 5y = 19
\end{cases}
\]

3. \[
\begin{cases}
x + y = 5 \\
x - y = -3
\end{cases}
\]

4. \[
\begin{cases}
x + y = 8 \\
x + y = 4
\end{cases}
\]

5. \[
\begin{cases}
2x - y = 8 \\
3x - 4y = 7
\end{cases}
\]

6. \[
\begin{cases}
2x - 4y = 14 \\
3x + y = 7
\end{cases}
\]

7. \[
\begin{cases}
x^2 + x = 0 \\
2yx + y = 0
\end{cases}
\]

8. \[
\begin{cases}
3y^2 + 6y = 0 \\
xy + x + 2 = 0
\end{cases}
\]

In Exercises 9–14, find all first and second partial derivatives of the function.

9. \( z = 4x^3 - 3y^2 \)

10. \( z = 2x^5 - y^3 \)

11. \( z = x^4 - \sqrt{xy} + 2y \)

12. \( z = 2x^2 - 3xy + y^2 \)

13. \( z = ye^{xy} \)

14. \( z = xe^{xy} \)

### EXERCISES 7.5

In Exercises 1–4, find any critical points and relative extrema of the function.

1. \( f(x, y) = x^2 - y^2 + 4x - 8y - 11 \)

2. \( f(x, y) = x^2 + y^2 + 2x - 6y + 6 \)

3. \( f(x, y) = \sqrt{x^2 + y^2 + 1} \)

4. \( f(x, y) = \sqrt{25 - (x - 2)^2 - y^2} \)

In Exercises 5–20, examine each function for relative extrema and saddle points.

5. \( f(x, y) = (x - 1)^2 + (y - 3)^2 \)

6. \( f(x, y) = 9 - (x - 3)^2 - (y + 2)^2 \)

7. \( f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3 \)

8. \( f(x, y) = -x^2 - 5y^2 + 8x - 10y - 13 \)

9. \( f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10 \)

10. \( f(x, y) = x^2 + 6xy + 10y^2 - 4y + 4 \)

11. \( f(x, y) = 3x^2 + 2y^2 - 12x - 4y + 7 \)

12. \( f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5 \)

13. \( f(x, y) = x^2 - y^2 + 4x - 4y - 8 \)

14. \( f(x, y) = x^2 - 3xy - 3y^2 \)

15. \( f(x, y) = xy \)

16. \( f(x, y) = 12x + 12y - xy - x^2 - y^3 \)
17. \( f(x, y) = (x^2 + 4y^2)e^{1-x^2-y^2} \)

18. \( f(x, y) = e^{-(x^2+y^2)} \)

19. \( f(x, y) = e^{xy} \)

20. \( f(x, y) = \frac{4x}{x^2 + y^2 + 1} \)

21. \( f_x(x_0, y_0) = 16 \)
   \( f_y(x_0, y_0) = 4 \)
   \( f_{xy}(x_0, y_0) = 8 \)

22. \( f_x(x_0, y_0) = -4 \)
   \( f_y(x_0, y_0) = -6 \)
   \( f_{xy}(x_0, y_0) = 3 \)

23. \( f_x(x_0, y_0) = -7 \)
   \( f_y(x_0, y_0) = 4 \)
   \( f_{xy}(x_0, y_0) = 9 \)

24. \( f_x(x_0, y_0) = 20 \)
   \( f_y(x_0, y_0) = 8 \)
   \( f_{xy}(x_0, y_0) = 9 \)

In Exercises 25–30, find the critical points and test for relative extrema. List the critical points for which the Second-Partials Test fails.

25. \( f(x, y) = (xy)^2 \)

26. \( f(x, y) = \sqrt{x^2 + y^2} \)

27. \( f(x, y) = x^3 + y^3 \)

28. \( f(x, y) = x^3 + y^3 - 3x^2 + 6y^2 + 3x + 12y + 7 \)

29. \( f(x, y) = x^{2/3} + y^{2/3} \)

30. \( f(x, y) = (x^2 + y^2)^{2/3} \)

In Exercises 31 and 32, find the critical points of the function and, from the form of the function, determine whether each critical point is a relative maximum or a relative minimum.

31. \( f(x, y, z) = (x - 1)^2 + (y + 3)^2 + z^2 \)

32. \( f(x, y, z) = 6 - [x(y + 2)(z - 1)]^2 \)

In Exercises 33–36, find three positive numbers \( x, y, \) and \( z \) that satisfy the given conditions.

33. The sum is 30 and the product is maximum.

34. The sum is 32 and \( P = xy^2z \) is maximum.

35. The sum is 30 and the sum of the squares is minimum.

36. The sum is 1 and the sum of the squares is minimum.

37. \( \text{Revenue} \) A company manufactures two products. The total revenue from \( x_1 \) units of product 1 and \( x_2 \) units of product 2 is given by

\[ R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2. \]

Find \( x_1 \) and \( x_2 \) so as to maximize the revenue.

38. \( \text{Revenue} \) A retail outlet sells two competitive products, the prices of which are \( p_1 \) and \( p_2 \). Find \( p_1 \) and \( p_2 \) so as to maximize the total revenue

\[ R = 500p_1 + 800p_2 + 1.5p_1p_2. \]
**SECTION 7.5 Extrema of Functions of Two Variables**

**Revenue** In Exercises 39 and 40, find $p_1$ and $p_2$ so as to maximize the total revenue $R = x_1 p_1 + x_2 p_2$ for a retail outlet that sells two competitive products with the given demand functions.

39. $x_1 = 1000 - 2p_1 + p_2$, $x_2 = 1500 + 2p_1 - 1.5p_2$

40. $x_1 = 1000 - 4p_1 + 2p_2$, $x_2 = 900 + 4p_1 - 3p_2$

41. **Profit** A corporation manufactures a high-performance automobile engine product at two locations. The cost functions for producing $x_1$ units at location 1 and $x_2$ units at location 2 are given by

$$C_1 = 0.05x_1^2 + 15x_1 + 5400$$

and

$$C_2 = 0.03x_2^2 + 15x_2 + 6100$$

respectively. The demand function for the product is given by

$$p = 225 - 0.4(x_1 + x_2)$$

and so the total revenue function is given by

$$R = (225 - 0.4(x_1 + x_2))(x_1 + x_2).$$

Find the production levels at the two locations that will maximize the profit

$$P = R - C_1 - C_2.$$

42. **Cost** The material for constructing the base of an open box costs 1.5 times as much as the material for constructing the sides. Find the dimensions of the box of largest volume that can be made for a fixed amount of money $C$.

43. **Volume** Find the dimensions of a rectangular package of largest volume that may be sent by a shipping company assuming that the sum of the length and the girth (perimeter of a cross section) cannot exceed 144 inches.

44. **Volume** Show that a rectangular box of given volume and minimum surface area is a cube.

45. **Hardy-Weinberg Law** Common blood types are determined genetically by the three alleles $A$, $B$, and $O$. (An allele is any of a group of possible mutational forms of a gene.) A person whose blood type is $AA$, $BB$, or $OO$ is homozygous. A person whose blood type is $AB$, $AO$, or $BO$ is heterozygous. The Hardy-Weinberg Law states that the proportion $P$ of heterozygous individuals in any given population is modeled by

$$P(p, q, r) = 2pq + 2pr + 2qr$$

where $p$ represents the percent of allele $A$ in the population, $q$ represents the percent of allele $B$ in the population, and $r$ represents the percent of allele $O$ in the population. Use the fact that $p + q + r = 1$ (the sum of the three must equal 100%) to show that the maximum proportion of heterozygous individuals in any population is $\frac{1}{2}$.

46. **Biology** A lake is to be stocked with smallmouth and largemouth bass. Let $x$ represent the number of smallmouth bass and let $y$ represent the number of largemouth bass in the lake. The weight of each fish is dependent on the population densities. After a six-month period, the weight of a single smallmouth bass is given by

$$W_1 = 3 - 0.002x - 0.005y$$

and the weight of a single largemouth bass is given by

$$W_2 = 4.5 - 0.003x - 0.004y.$$

Assuming that no fish die during the six-month period, how many smallmouth and largemouth bass should be stocked in the lake so that the total weight $T$ of bass in the lake is a maximum?

47. **Medicine** In order to treat a certain bacterial infection, a combination of two drugs is being tested. Studies have shown that the duration of the infection in laboratory tests can be modeled by

$$D(x, y) = x^2 + 2y^2 - 18x - 24y + 2xy + 120$$

where $x$ is the dose in hundreds of milligrams of the first drug and $y$ is the dose in hundreds of milligrams of the second drug. Determine the partial derivatives of $D$ will respect to $x$ and with respect to $y$. Find the amount of each drug necessary to minimize the duration of the infection.

**True or False?** In Exercises 48–51, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

48. If $d > 0$ and $f_x(a, b) < 0$, then $f(a, b)$ is a relative minimum.

49. A saddle point always occurs at a critical point.

50. If $f_x(x_0, y_0)$ is a relative maximum $(x_0, y_0, z_0)$, the $f_x(x_0, y_0) = f_x(x_0, y_0) = 0$.

51. The function

$$f(x, y) = \sqrt{x^2 + y^2}$$

has a relative maximum at the origin.
In Exercises 1–6, solve the system of linear equations.

1. \[
\begin{align*}
4x - 6y &= 3 \\
2x + 3y &= 2
\end{align*}
\]

2. \[
\begin{align*}
6x - 6y &= 5 \\
-3x - y &= 1
\end{align*}
\]

3. \[
\begin{align*}
5x - y &= 25 \\
x - 5y &= 15
\end{align*}
\]

4. \[
\begin{align*}
4x - 9y &= 5 \\
-x + 8y &= -2
\end{align*}
\]

5. \[
\begin{align*}
2x - y + z &= 3 \\
2x + 2y + z &= 4 \\
-x + 2y + 3z &= -1
\end{align*}
\]

6. \[
\begin{align*}
-x - 4y + 6z &= -2 \\
x - 3y - 3z &= 4 \\
3x + y + 3z &= 0
\end{align*}
\]

In Exercises 7–10, find all first partial derivatives.

7. \( f(x, y) = x^2y + xy^2 \)

8. \( f(x, y) = 25(xy + y^2)^2 \)

9. \( f(x, y, z) = x(x^2 - 2xy + yz) \)

10. \( f(x, y, z) = z(xy + xz + yz) \)

In Exercises 11–12, use Lagrange multipliers to find the given extremum. In each case, assume that \( x \) and \( y \) are positive.

<table>
<thead>
<tr>
<th>Objective Function</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Maximize ( f(x, y) = xy )</td>
<td>( x + y = 10 )</td>
</tr>
<tr>
<td>2. Maximize ( f(x, y) = xy )</td>
<td>( 2x + y = 4 )</td>
</tr>
<tr>
<td>3. Minimize ( f(x, y) = x^2 + y^2 )</td>
<td>( x + y - 4 = 0 )</td>
</tr>
<tr>
<td>4. Minimize ( f(x, y) = x^2 + y^2 )</td>
<td>( -2x - 4y + 5 = 0 )</td>
</tr>
<tr>
<td>5. Maximize ( f(x, y) = x^2 - y^2 )</td>
<td>( y - x^2 = 0 )</td>
</tr>
<tr>
<td>6. Minimize ( f(x, y) = x^2 - y^2 )</td>
<td>( x - 2y + 6 = 0 )</td>
</tr>
<tr>
<td>7. Maximize ( f(x, y) = 3x + xy + 3y )</td>
<td>( x + y = 25 )</td>
</tr>
<tr>
<td>8. Maximize ( f(x, y) = 3x + y + 10 )</td>
<td>( x^2y = 6 )</td>
</tr>
<tr>
<td>9. Maximize ( f(x, y) = \sqrt{x^2 - y^2} )</td>
<td>( x + y - 2 = 0 )</td>
</tr>
<tr>
<td>10. Minimize ( f(x, y) = \sqrt{x^2 + y^2} )</td>
<td>( 2x + 4y - 15 = 0 )</td>
</tr>
<tr>
<td>11. Maximize ( f(x, y) = e^{xy} )</td>
<td>( x^2 + y^2 - 8 = 0 )</td>
</tr>
<tr>
<td>12. Minimize ( f(x, y) = 2x + y )</td>
<td>( xy = 32 )</td>
</tr>
</tbody>
</table>
In Exercises 19 and 20, use Lagrange multipliers with the objective function
\[ f(x, y, z, w) = 2x^2 + y^2 + z^2 + 2w^2 \]
and with the given constraints to find the given extremum. In each case, assume that \( x, y, z, \) and \( w \) are nonnegative.

19. Maximize \( f(x, y, z, w) \)
   Constraint: \( 2x^2 + y^2 + z^2 + 2w^2 = 2 \)
20. Maximize \( f(x, y, z, w) \)
   Constraint: \( x + y + z + 2w = 4 \)

In Exercises 21–24, use Lagrange multipliers to find the given extremum of \( f \) subject to two constraints. In each case, assume that \( x, y, \) and \( z \) are nonnegative.

21. Maximize \( f(x, y, z) = xyz \)
   Constraints: \( x + y + z = 24, \ x - y - z = 12 \)
22. Minimize \( f(x, y, z) = x^2 + y^2 + z^2 \)
   Constraints: \( x + 2z = 4, \ x + y = 8 \)
23. Maximize \( f(x, y, z) = xyz \)
   Constraints: \( x^2 + z^2 = 5, \ x - 2y = 0 \)
24. Maximize \( f(x, y, z) = x^2 + yz \)
   Constraints: \( x + 2y = 6, \ x - 3z = 0 \)

In Exercises 25 and 26, use a spreadsheet to find the given extremum. In each case, assume that \( x, y, \) and \( z \) are nonnegative.

25. Maximize \( f(x, y, z) = xyz \)
   Constraints: \( x + 3y = 6, \ x - 2z = 0 \)
26. Minimize \( f(x, y, z) = x^2 + y^2 + z^2 \)
   Constraints: \( x + 2y = 8, \ x + z = 4 \)

In Exercises 27–30, find three positive numbers \( x, y, \) and \( z \) that satisfy the given conditions.

27. The sum is 120 and the product is maximum.
28. The sum is 120 and the sum of the squares is minimum.
29. The sum is 5 and the product is maximum.
30. The sum is 5 and the sum of the squares is minimum.

In Exercises 31–34, find the minimum distance from the curve or surface to the given point. (Hint: Start by minimizing the square of the distance.)

31. Line: \( x + 2y = 5, \ (0, 0) \)
   Minimize \( d^2 = x^2 + y^2 \)
32. Circle: \( (x - 4)^2 + y^2 = 4, \ (0, 10) \)
   Minimize \( d^2 = x^2 + (y - 10)^2 \)
33. Plane: \( x + y + z = 1, \ (2, 1, 1) \)
   Minimize \( d^2 = (x - 2)^2 + (y - 1)^2 + (z - 1)^2 \)
34. Cone: \( z = \sqrt{x^2 + y^2}, \ (4, 0, 0) \)
   Minimize \( d^2 = (x - 4)^2 + y^2 + z^2 \)
35. Volume Find the dimensions of the rectangular package of largest volume subject to the constraint that the sum of the length and the girth cannot exceed 108 inches (see figure). (Hint: Maximize \( V = xyz \) subject to the constraint \( x + 2y + 2z = 108 \).)

36. Cost In redecorating an office, the cost for new carpeting is five times the cost of wallpapering a wall. Find the dimensions of the largest office that can be redecorated for a fixed cost \( C \) (see figure). (Hint: Maximize \( V = xyz \) subject to \( 5xy + 2xz + 2yz = C \).)

37. Cost A cargo container (in the shape of a rectangular solid) must have a volume of 480 cubic feet. Use Lagrange multipliers to find the dimensions of the container of this size that has a minimum cost, if the bottom will cost $8 per square foot to construct and the sides and top will cost $3 per square foot to construct.

38. Cost A manufacturer has an order for 1000 units of fine paper that can be produced at two locations. Let \( x_1 \) and \( x_2 \) be the numbers of units produced at the two plants. Find the number of units that should be produced at each plant to minimize the cost if the cost function is given by
   \[ C = 0.25x_1^2 + 2x_1 + 0.05x_2^2 + 12x_2 \]

39. Cost A manufacturer has an order for 2000 units of all-terrain vehicle tires that can be produced at two locations. Let \( x_1 \) and \( x_2 \) be the numbers of units produced at the two plants. The cost function is modeled by
   \[ C = 0.25x_1^2 + 10x_1 + 0.15x_2^2 + 12x_2 \]
   Find the number of units that should be produced at each plant to minimize the cost.

40. Hardy-Weinberg Law Repeat Exercise 45 in Section 7.5 using Lagrange multipliers—that is, maximize
   \[ P(p, q, r) = 2pq + 2pr + 2qr \]
   subject to the constraint
   \[ p + q + r = 1 \]

41. Least-Cost \( Ri \) party is given by
   \[ f(x, y) = 10 \]
   where \( x \) is the number of units of capital and \( y \) is the number of units of labor. (a) Find the number of units of capital and labor. (b) Show that the marginal product of labor is greater than the marginal product of capital. (c) This proportion is called the labor-capital ratio. (d) This proportion is called the labor-capital ratio. (e) This proportion is called the labor-capital ratio.

42. Least-Cost \( Ru \)
   function given by
   \[ f(x, y) = 10 \]
   for \( x, y \) is the number of units of capital and labor. (a) Find the maximum profit. (b) Find the maximum profit. (c) Use the maximum profit. (d) Use the maximum profit. (e) Use the maximum profit.

43. Production \( 1 \)
   function given by
   \[ f(x, y) = 10 \]

44. Production \( 2 \)
   function given by
   \[ f(x, y) = 10 \]

45. Biology A mixture of which \( g \) of salt is used. Find the number of units of "salt" that should be used.

46. Biology Repeat Exercise 45 in Section 7.5 using Lagrange multipliers—that is, maximize
   \[ P(p, q, r) = 2pq + 2pr + 2qr \]
   subject to the constraint
   \[ p + q + r = 1 \]
41. Least-Cost Rule  The production function for a company is given by

\[ f(x, y) = 100x^{0.25}y^{0.75} \]

where \( x \) is the number of units of labor and \( y \) is the number of units of capital. Suppose that labor costs \$48 per unit, capital costs \$36 per unit, and management sets a production goal of 20,000 units.

(a) Find the numbers of units of labor and capital needed to meet the production goal while minimizing the cost.

(b) Show that the conditions of part (a) are met when

\[
\text{Marginal productivity of labor} \cdot \text{unit price of labor} = \text{Marginal productivity of capital} \cdot \text{unit price of capital}
\]

This proportion is called the Least-Cost Rule (or Equimarginal Rule).

42. Least-Cost Rule  Repeat Exercise 41 for the production function given by

\[ f(x, y) = 100x^{0.25}y^{0.75} \]

43. Production  The production function for a company is given by

\[ f(x, y) = 100x^{0.25}y^{0.75} \]

where \( x \) is the number of units of labor and \( y \) is the number of units of capital. Suppose that labor costs \$48 per unit and capital costs \$36 per unit. The total cost of labor and capital is limited to \$100,000.

(a) Find the maximum production level for this manufacturer.

(b) Find the marginal productivity of money.

(c) Use the marginal productivity of money to find the maximum number of units that can be produced if \$125,000 is available for labor and capital.

44. Production  Repeat Exercise 43 for the production function given by

\[ f(x, y) = 100x^{0.25}y^{0.75} \]

45. Biology  A microbiologist must prepare a culture medium in which to grow a certain type of bacteria. The percent of salt contained in this medium is given by

\[ S = 12xyz \]

where \( x, y, \) and \( z \) are the nutrient solutions to be mixed in the medium. For the bacteria to grow, the medium must be 13% salt. Nutrient solutions \( x, y, \) and \( z \) cost \$1, \$2, and \$3 per liter, respectively. How much of each nutrient solution should be used to minimize the cost of the culture medium?

46. Biology  Repeat Exercise 45 for a salt-content model of

\[ S = 0.01x^2y^2z^2. \]

47. Construction  A rancher plans to use an existing stone wall and the side of a barn as a boundary for two adjacent rectangular corrals. Fencing for the perimeter costs \$10 per foot. To separate the corrals, a fence that costs \$4 per foot will divide the region. The total area of the two corrals is to be 6000 square feet.

(a) Use Lagrange multipliers to find the dimensions that will minimize the cost of the fencing.

(b) What is the minimum cost?

48. Area  Use Lagrange multipliers to show that the maximum area of a rectangle with dimensions \( x \) and \( y \) and a given perimeter \( P \) is \( \frac{1}{16}P^2 \).

49. Investment Strategy  An investor is considering three different stocks in which to invest \$300,000. The average annual dividends for the stocks are

- General Mills (G) 2.5%
- PepsiCo, Inc. (P) 1.4%
- Sara Lee (S) 3.1%

The amount invested in PepsiCo, Inc. must follow the equation

\[ 3000(G) - 3000(S) + P^2 = 0. \]

How much should be invested in each stock to yield a maximum of dividends?

50. Investment Strategy  An investor is considering three different stocks in which to invest \$20,000. The average annual dividends for the stocks are

- General Motors (G) 5.2%
- Campbell Soup (C) 2.7%
- Kellogg Co. (K) 3.2%

The amount invested in Campbell Soup must follow the equation

\[ 1000(K) - 1000(G) + C^2 = 0. \]

How much should be invested in each stock to yield a maximum of dividends?

51. Research Project  Use your school's library, the Internet, or some other reference source to write a paper about two different types of available investment options. Find examples of each type and find the data about their dividends for the past 10 years. What are the similarities and differences between the two types?
The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–12, evaluate the definite integral.

1. \( \int_0^1 \frac{x}{x} \, dx \)
2. \( \int_0^3 3 \, dy \)
3. \( \int_0^4 2x^2 \, dx \)
4. \( \int_0^1 2x^3 \, dx \)
5. \( \int_1^2 (x^3 - 2x + 4) \, dx \)
6. \( \int_0^1 (4 - y^2) \, dy \)
7. \( \int_1^2 \frac{2}{\sqrt{x}} \, dx \)
8. \( \int_1^4 \frac{\sqrt{y}}{x} \, dx \)
9. \( \int_0^2 \frac{2x}{x^2 + 1} \, dx \)
10. \( \int_1^2 \frac{1}{y - 1} \, dy \)
11. \( \int_0^2 xe^{x^2} \, dx \)
12. \( \int_0^1 e^{-2y} \, dy \)

In Exercises 13–16, sketch the region bounded by the graphs of the equations.

13. \( y = x, \ y = 0, \ x = 3 \)
14. \( y = x, \ y = 3, \ x = 0 \)
15. \( y = 4 - x^2, \ y = 0, \ x = 0 \)
16. \( y = x^2, \ y = 4x \)

In Exercises 1–10, evaluate the partial integral.

1. \( \int_0^1 (2x - y) \, dy \)
2. \( \int_0^2 \frac{y}{x} \, dx \)
3. \( \int_0^2 \frac{y}{x} \, dx \)
4. \( \int_0^1 y \, dx \)
5. \( \int_1^2 x^2y \, dy \)
6. \( \int_1^2 (x^3 + y^2) \, dy \)
7. \( \int_0^1 \frac{y \ln x}{x} \, dx \)
8. \( \int_1^2 \frac{x^3}{x + 1} \, dx \)
9. \( \int_0^1 ye^{-y/4} \, dx \)
10. \( \int_0^3 \frac{xy}{x^2 + 1} \, dx \)

In Exercises 11–24, evaluate the double integral.

11. \( \int_0^3 \int_0^1 (x - y) \, dy \, dx \)
12. \( \int_0^3 \int_0^2 (6 - x^2) \, dy \, dx \)
13. \( \int_0^3 \int_0^1 xy \, dy \, dx \)
14. \( \int_0^3 \int_0^x \sqrt{x^2 + 4} \, dy \, dx \)
15. \( \int_0^3 \int_0^1 (x + y) \, dx \, dy \)
16. \( \int_0^3 \int_0^{3y^2} 3y \, dx \, dy \)
17. \( \int_0^1 \int_0^1 (x^2 - 2y^2 + 1) \, dx \, dy \)
18. \( \int_0^1 \int_0^2 (1 + 2x^2 + 2y^2) \, dx \, dy \)
19. \( \int_0^1 \int_0^2 \sqrt{1-x^2} \, dx \, dy \)
20. \( \int_0^1 \int_0^2 \frac{2}{x^2 + 4} \, dx \, dy \)
21. \( \int_0^3 \int_0^1 x^2 \, dy \, dx \)
22. \( \int_0^3 \int_0^1 (x^3 + y^2) \, dy \, dx \)
23. \( \int_0^2 \int_0^1 e^{-(x+y)^2} \, dy \, dx \)
24. \( \int_0^2 \int_0^{3y^2} xy e^{-(x+y)^2} \, dx \, dy \)

In Exercises 25–32, sketch the region bounded by the graphs of the equations.

25. \( \int_0^2 \int_0^3 dy \, dx \)
26. \( \int_0^2 \int_0^4 dx \, dy \)
27. \( \int_0^2 \int_0^{2y} dx \, dy \)
28. \( \int_0^2 \int_0^{\sqrt{2}} dx \, dy \)
29. \( \int_0^2 \int_0^4 \sqrt{x} \, dy \, dx \)
30. \( \int_0^2 \int_0^{2y} dx \, dy \)
31. \( \int_0^2 \int_0^{\sqrt{x^2}} dx \, dy \)
32. \( \int_0^2 \int_0^{x^2} dx \, dy \)

In Exercises 33 and 34, is necessary to change the order of integration.

33. \( \int_0^3 \int_y^3 e^{x^2} \, dx \, dy \)
34. \( \int_0^3 \int_y e^{x^2} \, dx \, dy \)

In Exercises 35–40, use the specified region.

35. \( y = x \)
36. \( y = 4 \)
37. \( y = 4 \)
In Exercises 25–32, sketch the region R whose area is given by the
double integral. Then change the order of integration and show
that both orders yield the same area.

25. \[ \int_0^1 \int_0^2 dy \, dx \]
26. \[ \int_0^2 \int_1^2 dx \, dy \]
27. \[ \int_0^1 \int_{\sqrt{2}}^2 dx \, dy \]
28. \[ \int_0^4 \int_0^2 dy \, dx \]
29. \[ \int_0^1 \int_0^{1/2} dy \, dx \]
30. \[ \int_0^2 \int_0^{x/2} dx \, dy \]
31. \[ \int_0^1 \int_0^{\sqrt{y}} dx \, dy \]
32. \[ \int_{-2}^2 \int_0^{4-y^2} dx \, dy \]

In Exercises 33 and 34, evaluate the double integral. Note that it
is necessary to change the order of integration.

33. \[ \int_0^3 \int_0^3 e^{x^2} \, dx \, dy \]
34. \[ \int_0^2 \int_0^2 e^{-x^2} \, dx \, dy \]

In Exercises 35–40, use a double integral to find the area of the
specified region.

35. \[ y = 4 - x^2 \]
36. \[ y = \sqrt{x + \sqrt{y}} = 2 \]

\[ y \]
\[ \begin{array}{c}
\text{(1, 3)} \quad \text{(3, 3)} \\
\end{array} \]

\[ (1, 1) \quad (3, 1) \]

37. \[ y = 4 - x^2 \]
38. \[ y = x + 2 \]

In Exercises 41–46, use a double integral to find the area of the
region bounded by the graphs of the equations.

41. \[ y = 25 - x^2, \quad y = 0 \]
42. \[ y = x^{3/2}, \quad y = x \]
43. \[ 5x - 2y = 0, \quad x + y = 3, \quad y = 0 \]
44. \[ xy = 9, \quad x = y, \quad y = 0, \quad x = 9 \]
45. \[ y = x, \quad y = 2x, \quad x = 2 \]
46. \[ y = x^2 + 2x + 1, \quad y = 3(x + 1) \]

In Exercises 47–54, use a symbolic integration utility to evaluate
the double integral.

47. \[ \int_0^1 \int_0^2 e^{-x^2 - y^2} \, dx \, dy \]
48. \[ \int_0^2 \int_0^{x^3 + 3y^2} (x^3 + 3y^2) \, dy \, dx \]
49. \[ \int_0^1 \int_0^1 e^{x^2} \, dx \, dy \]
50. \[ \int_0^1 \int_0^1 \ln(x + y) \, dx \, dy \]
51. \[ \int_0^1 \int_0^x \sqrt{1 - x^2} \, dy \, dx \]
52. \[ \int_0^1 \int_0^x \sqrt{x + \sqrt{1 + x}} \, dy \, dx \]
53. \[ \int_0^1 \int_0^{4-y^2/4} \frac{xy}{x^2 + y^2 + 1} \, dy \, dx \]
54. \[ \int_0^1 \int_0^{(y+1)(x+1)} \frac{2}{(x+1)(y+1)} \, dx \, dy \]

**True or False?** In Exercises 55 and 56, determine whether
the statement is true or false. If it is false, explain why or give an
example that shows it is false.

55. Changing the order of integration will sometimes change
the value of a double integral.
56. \[ \int_{-1}^1 \int_{-1}^1 x \, dy \, dx = \int_{-1}^1 \int_{-1}^1 y \, dx \, dy \]
The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, sketch the region that is described.
1. $0 \leq x \leq 2$, $0 \leq y \leq 1$
2. $1 \leq x \leq 3$, $2 \leq y \leq 3$
3. $0 \leq x \leq 4$, $0 \leq y \leq 2x - 1$
4. $0 \leq x \leq 2$, $0 \leq y \leq x^2$

In Exercises 5–10, evaluate the double integral.
5. $\int_0^1 \int_0^1 dy \, dx$
6. $\int_0^1 \int_1^3 dx \, dy$
7. $\int_0^1 \int_0^x x \, dy \, dx$
8. $\int_0^1 \int_0^1 y \, dx \, dy$
9. $\int_0^3 \int_1^{e^2} 2 \, dy \, dx$
10. $\int_0^1 \int_0^{-x^2 + 2} dy \, dx$

Exercise 7.9

In Exercises 1–8, sketch the region of integration and evaluate the double integral.
1. $\int_0^2 \int_0^{3x + 4y} dy \, dx$
2. $\int_0^3 \int_0^{2x + 6y} dx \, dy$
3. $\int_0^1 \int_0^{\sqrt{y}} x^2 \, dy \, dx$
4. $\int_0^6 \int_0^{x+y} dx \, dy$
5. $\int_0^1 \int_0^{\sqrt{1-x^2}} dy \, dx$
6. $\int_0^2 \int_0^{4-x^2} xy^2 \, dy \, dx$
7. $\int_{-a}^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dy \, dx$
8. $\int_0^\infty \int_0^{\infty} \frac{1}{\ln y} \, dy \, dx$

In Exercises 9–12, set up the integral for both orders of integration and use the more convenient order to evaluate the integral over the region $R$.
9. $\int_R \int_R xy \, dA$
   $R$: rectangle with vertices at $(0, 0), (0, 5), (3, 5), (3, 0)$
10. $\int_R \int_R x \, dA$
    $R$: semicircle bounded by $y = \sqrt{25 - x^2}$ and $y = 0$
11. $\int_R \int_R \frac{y}{x^2 + y^2} \, dA$
    $R$: triangle bounded by $y = x, y = 2x, x = 2$
12. $\int_R \int_R \frac{y}{1 + x^2} \, dA$
    $R$: region bounded by $y = 0, y = \sqrt{x}, x = 4$

In Exercises 13 and 14, evaluate the double integral. Note that it is necessary to change the order of integration.
13. $\int_0^{\frac{1}{\sqrt{2}}} \int_y^{1/2} e^{-x^2} \, dy \, dx$
14. $\int_0^{\infty} \int_0^{10} \frac{1}{\ln y} \, dy \, dx$

In Exercises 15–26, use a double integral to find the volume of the specified solid.
15. $z = \frac{y}{2}$
16. $z = 6 - 2y$
17. $z = 8 - x - y$
18. $z = x$
19. $2x + 3y + 4z = 12$
20. $x = 1$
21. $x = y$
22. $x = z$
23. $z = 4 - x^2 - y^2$
24. $x^2 + y^2 = 1$
25. $x^2 + z^2 = 1$
26. $z = xy, z = 0$
27. $z = x, z = 0$
28. $z = x, z = 0$
29. \( z = x^2, \quad z = 0, \quad x = 0, \quad x = 2, \quad y = 0, \quad y = 4 \)
30. \( z = x + y, \quad x^2 + y^2 = 4 \) (first octant)

31. **Population Density** The population density (in people per square mile) for a coastal town can be modeled by
   
   \[ f(x, y) = \frac{120,000}{(x + y)^3} \]
   
   where \( x \) and \( y \) are measured in miles. What is the population inside the rectangular area defined by the vertices \((0, 0), (2, 0), (0, 2), \) and \((2, 2)\)?

32. **Population Density** The population density (in people per square mile) for a coastal town on an island can be modeled by
   
   \[ f(x, y) = \frac{5000x^2y}{1 + 2x^2} \]
   
   where \( x \) and \( y \) are measured in miles. What is the population inside the rectangular area defined by the vertices \((0, 0), (4, 0), (0, -2), \) and \((4, -2)\)?

In Exercises 33–36, find the average value of \( f(x, y) \) over the region \( R \).

\[
\begin{align*}
33. \quad & f(x, y) = x \\
& \text{Rectangle with vertices } (0, 0), (4, 0), (4, 2), (0, 2) \\
34. \quad & f(x, y) = xy \\
& \text{Rectangle with vertices } (0, 0), (4, 0), (4, 2), (0, 2) \\
35. \quad & f(x, y) = x^2 + y^2 \\
& \text{Square with vertices } (0, 0), (2, 0), (2, 2), (0, 2) \\
36. \quad & f(x, y) = e^{x+y} \\
& \text{Triangle with vertices } (0, 0), (0, 1), (1, 1)
\end{align*}
\]

37. **Average Revenue** A company sells two products whose demand functions are given by
   
   \[ x_1 = 500 - 3p_1 \quad \text{and} \quad x_2 = 750 - 2.4p_2. \]
   
   So, the total revenue is given by
   
   \( R = x_1p_1 + x_2p_2. \)
   
   Estimate the average revenue if the price \( p_1 \) varies between \$50 and \$75 and the price \( p_2 \) varies between \$100 and \$150.

38. **Average Weekly Profit** A firm's weekly profit in marketing two products is given by
   
   \[ P = 192x_1 + 576x_2 - x_1^2 - 5x_2^2 - 2x_1x_2 - 5000 \]
   
   where \( x_1 \) and \( x_2 \) represent the numbers of units of each product sold weekly. Estimate the average weekly profit if \( x_1 \) varies between 40 and 50 units and \( x_2 \) varies between 45 and 50 units.