PREREQUISITE REVIEW 7.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, evaluate the function when $x = -3$.

1. $f(x) = 5 - 2x$

2. $f(x) = -x^2 + 4x + 5$

3. $y = \sqrt{4x^2 - 3x + 4}$

4. $y = \sqrt[3]{34 - 4x + 2x^2}$

In Exercises 5–8, find the domain of the function.

5. $f(x) = 5x^2 + 3x - 2$

6. $g(x) = \dfrac{1}{2x} - \dfrac{2}{x+3}$

7. $h(y) = \sqrt{y - 5}$

8. $f(y) = \sqrt{y^2 - 5}$

In Exercises 9 and 10, evaluate the expression.

9. $(476)^{0.65}$

10. $(251)^{0.35}$

EXERCISES 7.3

In Exercises 1–14, find the function values.

1. $f(x, y) = \dfrac{x}{y}$

(a) $f(3, 2)$ (b) $f(-1, 4)$
(c) $f(30, 5)$ (d) $f(5, y)$
(e) $f(x, 2)$ (f) $f(5, t)$

2. $f(x, y) = 4 - x^2 - 4y^2$

(a) $f(0, 0)$ (b) $f(0, 1)$
(c) $f(2, 3)$ (d) $f(1, y)$
(e) $f(x, 0)$ (f) $f(t, 1)$

3. $f(x, y) = xe^y$

(a) $f(5, 0)$ (b) $f(3, 2)$
(c) $f(2, -1)$ (d) $f(5, y)$
(e) $f(x, 2)$ (f) $f(t, t)$

4. $g(x, y) = \ln|x + y|$

(a) $g(2, 3)$ (b) $g(5, 6)$
(c) $g(e, 0)$ (d) $g(0, 1)$
(e) $g(2, -3)$ (f) $g(e, e)$

5. $h(x, y, z) = \dfrac{xy}{z}$

(a) $h(2, 3, 9)$ (b) $h(1, 0, 1)$

6. $f(x, y, z) = \sqrt{x + y + z}$

(a) $f(0, 5, 4)$ (b) $f(6, 8, -3)$

7. $V(r, h) = \pi r^2 h$

(a) $V(3, 10)$ (b) $V(5, 2)$

8. $F(r, N) = 500\left(1 + \dfrac{r}{12}\right)^N$

(a) $F(0.09, 60)$ (b) $F(0.14, 240)$

9. $A(P, r, t) = P\left[\left(1 + \dfrac{r}{12}\right)^{12t} - 1\right]\left(1 + \dfrac{12}{r}\right)$

(a) $A(100, 0.10, 10)$ (b) $A(275, 0.0925, 40)$

10. $A(P, r, t) = Pe^{rt}$

(a) $A(500, 0.10, 5)$ (b) $A(1500, 0.12, 20)$

11. $f(x, y) = \displaystyle\int_x^y (2t - 3)\,dt$

(a) $f(1, 2)$ (b) $f(1, 4)$

12. $g(x, y) = \displaystyle\int_x^y \frac{1}{t}\,dt$

(a) $g(4, 1)$ (b) $g(6, 3)$

13. $f(x, y) = x^2 - 2y$

(a) $f(x + \Delta x, y)$

(b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

14. $f(x, y) = 3xy + y^2$

(a) $f(x + \Delta x, y)$

(b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

In Exercises 15–18, des
plane that corresponds
the range of the functic

15. $f(x, y) = \sqrt{16 - x}$

16. $f(x, y) = x^2 + y^2$ -

17. $f(x, y) = e^{x/y}$

18. $f(x, y) = \ln(x + y)$

In Exercises 19–28, des
plane that corresponds

19. $f(x, y) = \sqrt{9 - 9x}$

20. $f(x, y) = \sqrt{x^2 + y}$

21. $f(x, y) = \dfrac{x}{y}$

22. $f(x, y) = \dfrac{4y}{x - 1}$

23. $f(x, y) = \dfrac{1}{xy}$

24. $g(x, y) = \dfrac{1}{x - y}$

25. $h(x, y) = x\sqrt{y}$

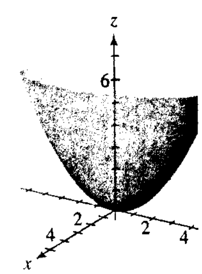
26. $f(x, y) = \sqrt{xy}$

27. $g(x, y) = \ln(4 - x$

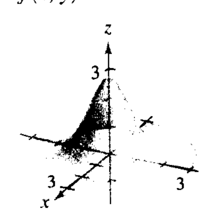
28. $f(x, y) = ye^{1/x}$

In Exercises 29–32, mat
the contour maps. [The

29. $f(x, y) = x^2 + \dfrac{y^2}{4}$



31. $f(x, y) = e^{1 - x^2 - y^2}$



earlier sections. You will

) $V(5, 2)$

) $F(0.14, 240)$

$\Big]\left(1 + \frac{12}{r}\right)$

) $A(275, 0.0925, 40)$

) $A(1500, 0.12, 20)$

$f(1, 4)$

$g(6, 3)$

In Exercises 15–18, describe the region R in the xy-coordinate plane that corresponds to the domain of the function, and find the range of the function.

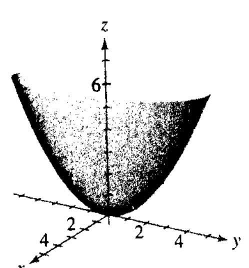
15. $f(x, y) = \sqrt{16 - x^2 - y^2}$
16. $f(x, y) = x^2 + y^2 - 1$
17. $f(x, y) = e^{x/y}$
18. $f(x, y) = \ln(x + y)$

In Exercises 19–28, describe the region R in the xy-coordinate plane that corresponds to the domain of the function.

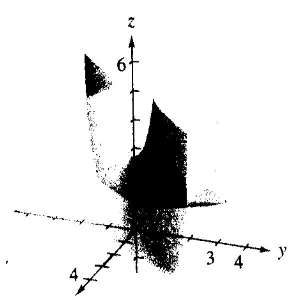
19. $f(x, y) = \sqrt{9 - 9x^2 - y^2}$
20. $f(x, y) = \sqrt{x^2 + y^2 - 1}$
21. $f(x, y) = \dfrac{x}{y}$
22. $f(x, y) = \dfrac{4y}{x - 1}$
23. $f(x, y) = \dfrac{1}{xy}$
24. $g(x, y) = \dfrac{1}{x - y}$
25. $h(x, y) = x\sqrt{y}$
26. $f(x, y) = \sqrt{xy}$
27. $g(x, y) = \ln(4 - x - y)$
28. $f(x, y) = ye^{1/x}$

In Exercises 29–32, match the graph of the surface with one of the contour maps. [The contour maps are labeled (a)–(d).]

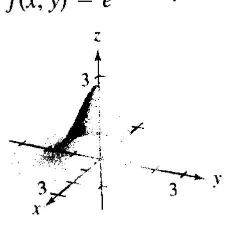
29. $f(x, y) = x^2 + \dfrac{y^2}{4}$



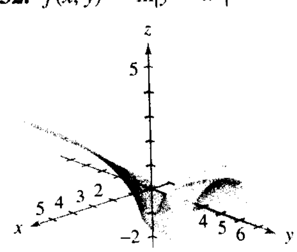
30. $f(x, y) = e^{1-x^2+y^2}$

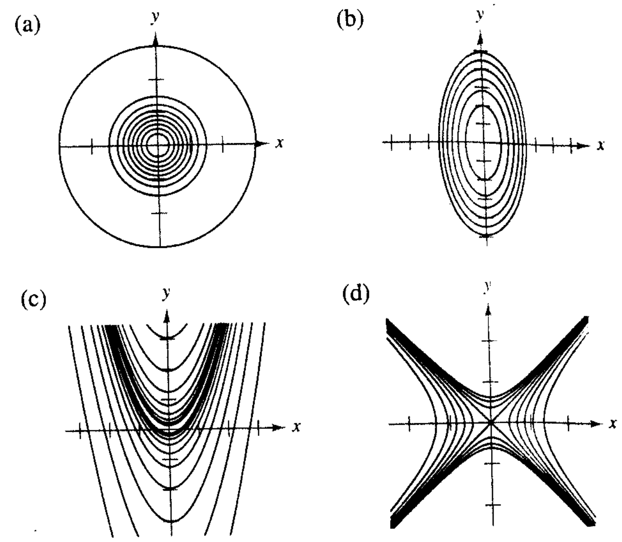


31. $f(x, y) = e^{1-x^2-y^2}$



32. $f(x, y) = \ln|y - x^2|$





In Exercises 33–40, describe the level curves of the function. Sketch the level curves for the given c-values.

Function	*c-Values*
33. $z = x + y$	$c = -1, 0, 2, 4$
34. $z = 6 - 2x - 3y$	$c = 0, 2, 4, 6, 8, 10$
35. $z = \sqrt{16 - x^2 - y^2}$	$c = 0, 1, 2, 3, 4$
36. $f(x, y) = x^2 + y^2$	$c = 0, 2, 4, 6, 8$
37. $f(x, y) = xy$	$c = \pm 1, \pm 2, \ldots, \pm 6$
38. $z = e^{xy}$	$c = 1, 2, 3, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$
39. $f(x, y) = \dfrac{x}{x^2 + y^2}$	$c = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$
40. $f(x, y) = \ln(x - y)$	$c = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$

41. ***Cobb-Douglas Production Function*** A manufacturer estimates the Cobb-Douglas production function to be given by

$$f(x, y) = 100x^{0.75}y^{0.25}.$$

Estimate the production levels when $x = 1500$ and $y = 1000$.

42. ***Cobb-Douglas Production Function*** Use the Cobb-Douglas production function (Example 5) to show that if both the number of units of labor and the number of units of capital are doubled, the production level is also doubled.

43. ***Cost*** A company manufactures two types of woodburning stoves: a freestanding model and a fireplace-insert model. The cost function for producing x freestanding stoves and y fireplace-insert stoves is given by

$$C(x, y) = 27\sqrt{xy} + 195x + 215y + 980.$$

Find the cost when $x = 80$ and $y = 20$.

44. *Forestry* The *Doyle Log Rule* is one of several methods used to determine the lumber yield of a log in board-feet in terms of its diameter d in inches and its length L in feet. The number of board-feet is given by

$$N(d, L) = \left(\frac{d - 4}{4}\right)^2 L.$$

(a) Find the number of board-feet of lumber in a log with a diameter of 22 inches and a length of 12 feet.

(b) Find $N(30, 12)$.

45. *Profit* A sporting goods manufacturer produces regulation soccer balls at two plants. The costs of producing x_1 units at location 1 and x_2 units at location 2 are given by

$$C_1(x_1) = 0.02x_1^2 + 4x_1 + 500$$

and

$$C_2(x_2) = 0.05x_2^2 + 4x_2 + 275$$

respectively. If the product sells for \$45 per unit, then the profit function for the product is given by

$$P(x_1, x_2) = 45(x_1 + x_2) - C_1(x_1) - C_2(x_2).$$

Find (a) $P(250, 150)$ and (b) $P(300, 200)$.

46. *Consumer Awareness* The average amount of time that a customer waits in line for service is given by

$$W(x, y) = \frac{1}{x - y}, \qquad y < x$$

where y is the average arrival rate and x is the average service rate (x and y are measured in the number of customers per hour). Evaluate W at each point.

(a) $(15, 10)$ (b) $(12, 9)$ (c) $(12, 6)$ (d) $(4, 2)$

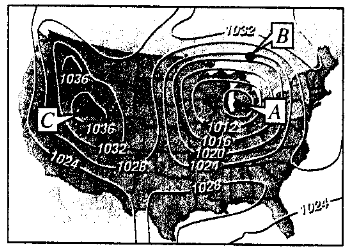
47. *Investment* In 2004, an investment of \$1000 was made in a bond earning 10% compounded annually. The investor pays tax at rate R, and the annual rate of inflation is I. In the year 2014, the value V of the bond in constant 2004 dollars is given by

$$V(I, R) = 1000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^{10}.$$

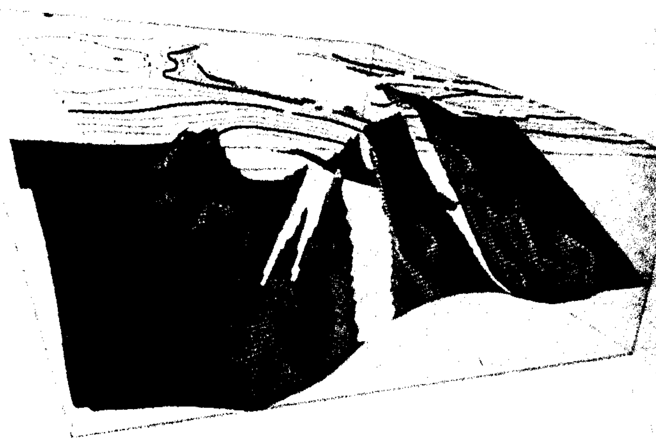
Use this function of two variables to complete the table.

		Inflation rate, I		
		0.00	0.03	0.05
Tax rate, R	0.00			
	0.28			
	0.35			

48. *Meteorology* Meteorologists measure the atmospheric pressure in millibars. From these observations they create weather maps on which the curves of equal atmospheric pressure (isobars) are drawn (see figure). On the map, the closer the isobars the higher the wind speed. Match points A, B, and C with (a) highest pressure, (b) lowest pressure, and (c) highest wind velocity.



49. *Geology: A Contour Map* The contour map below represents color-coded seismic amplitudes of a fault horizon and a projected contour map, which is used in earthquake studies. *(Source: Adapted from Shipman/Wilson/Todd, An Introduction to Physical Science, Tenth Edition)*



Shipman, *An Introduction to Physical Science* 10/e, 2003, Houghton Mifflin Company

(a) Discuss the use of color to represent the level curves.

(b) Do the level curves correspond to equally spaced amplitudes? Explain your reasoning.

50. *Earnings per Share* The earnings per share for Starbucks Corporation from 1995 through 2003 can be modeled by

$$z = 0.265x - 0.209y + 0.033$$

where x is sales (in billions of dollars) and y is the shareholder's equity (in billions of dollars). *(Source: Starbucks Corporation)*

(a) Find the earnings per share when $x = 8$ and $y = 5$.

(b) Which of the two variables in this model has the greater influence on the earnings per share? Explain your reasoning.

7.

Functions of

Real-life applicati
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f with respect to o
stant. This process
a **partial derivati**
tives as it has inde

Partial Deriva

If $z = f(x, y)$, t
and y are the f

$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0}$

$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0}$

EXAMPLE 1

Find $\partial z/\partial x$ and ∂z

SOLUTION

$\frac{\partial z}{\partial x} = 3 - 2xy$

$\frac{\partial z}{\partial y} = -2x^2y$ -

TRY IT 1

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ f

PREREQUISITE REVIEW 7.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–8, find the derivative of the function.

1. $f(x) = \sqrt{x^2 + 3}$

2. $g(x) = (3 - x^2)^3$

3. $g(t) = te^{2t+1}$

4. $f(x) = e^{2x}\sqrt{1 - e^{2x}}$

5. $f(x) = \ln(3 - 2x)$

6. $u(t) = \ln\sqrt{t^3 - 6t}$

7. $g(x) = \dfrac{5x^2}{(4x - 1)^2}$

8. $f(x) = \dfrac{(x + 2)^3}{(x^2 - 9)^2}$

In Exercises 9 and 10, evaluate the derivative at the point (2, 4).

9. $f(x) = x^2e^{x-2}$

10. $g(x) = x\sqrt{x^2 - x + 2}$

EXERCISES 7.4

In Exercises 1–14, find the first partial derivatives with respect to x and with respect to y.

1. $f(x, y) = 2x - 3y + 5$

2. $f(x, y) = x^2 - 3y^2 + 7$

3. $f(x, y) = 5\sqrt{x} - 6y^2$

4. $f(x, y) = x^{-1/2} + 4y^{3/2}$

5. $f(x, y) = \dfrac{x}{y}$

6. $z = x\sqrt{y}$

7. $f(x, y) = \sqrt{x^2 + y^2}$

8. $f(x, y) = \dfrac{xy}{x^2 + y^2}$

9. $z = x^2e^{2y}$

10. $z = xe^{x+y}$

11. $h(x, y) = e^{-(x^2+y^2)}$

12. $g(x, y) = e^{x/y}$

13. $z = \ln\dfrac{x - y}{(x + y)^2}$

14. $g(x, y) = \ln\sqrt{x^2 + y^2}$

In Exercises 15–20, let $f(x, y) = 3x^2ye^{x-y}$ and $g(x, y) = 3xy^2e^{y-x}$. Find each of the following.

15. $f_x(x, y)$

16. $f_y(x, y)$

17. $g_x(x, y)$

18. $g_y(x, y)$

19. $f_x(1, 1)$

20. $g_x(-2, -2)$

In Exercises 21–28, evaluate f_x and f_y at the point.

	Function	*Point*
21.	$f(x, y) = 3x^2 + xy - y^2$	$(2, 1)$
22.	$f(x, y) = x^2 - 3xy + y^2$	$(1, -1)$
23.	$f(x, y) = e^{3xy}$	$(0, 4)$
24.	$f(x, y) = e^xy^2$	$(0, 2)$
25.	$f(x, y) = \dfrac{xy}{x - y}$	$(2, -2)$
26.	$f(x, y) = \dfrac{4xy}{\sqrt{x^2 + y^2}}$	$(1, 0)$
27.	$f(x, y) = \ln(x^2 + y^2)$	$(1, 0)$
28.	$f(x, y) = \ln\sqrt{xy}$	$(-1, -1)$

In Exercises 29–32, find the first partial derivatives with respect to x, y, and z.

29. $w = 3x^2y - 5xyz + 10yz^2$

30. $w = \sqrt{x^2 + y^2 + z^2}$

31. $w = \dfrac{xy}{x + y + z}$

32. $w = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

In Exercises 33–38, evaluate w_x, w_y, and w_z at the point.

	Function	*Point*
33.	$w = \sqrt{x^2 + y^2 + z^2}$	$(2, -1, 2)$
34.	$w = \dfrac{xy}{x + y + z}$	$(1, 2, 0)$
35.	$w = \ln\sqrt{x^2 + y^2 + z^2}$	$(3, 0, 4)$
36.	$w = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$	$(0, 0, 0)$
37.	$w = 2xz^2 + 3xyz - 6y^2z$	$(1, -1, 2)$
38.	$w = xye^{z^2}$	$(2, 1, 0)$

In Exercises 39–42, find values of x and y such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.

39. $f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$

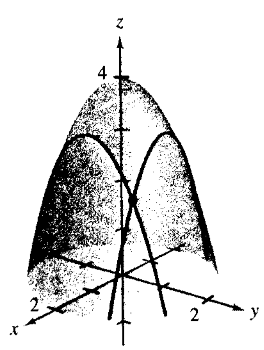
40. $f(x, y) = 3x^3 - 12xy + y^3$

41. $f(x, y) = \frac{1}{x} + \frac{1}{y} + xy$

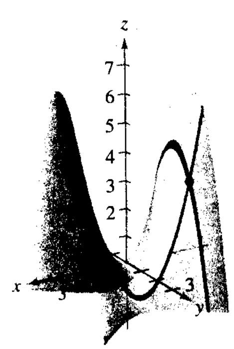
42. $f(x, y) = \ln(x^2 + y^2 + 1)$

In Exercises 43–50, find the slope of the surface at the given point in (a) the x-direction and (b) the y-direction.

	Function	*Point*
43.	$z = 2x - 3y + 5$	$(2, 1, 6)$
44.	$z = xy$	$(1, 2, 2)$
45.	$z = x^2 - 9y^2$	$(3, 1, 0)$
46.	$z = x^2 + 4y^2$	$(2, 1, 8)$
47.	$z = \sqrt{25 - x^2 - y^2}$	$(3, 0, 4)$
48.	$z = \frac{x}{y}$	$(3, 1, 3)$
49.	$z = 4 - x^2 - y^2$	$(1, 1, 2)$



50. $z = x^2 - y^2$ $(-2, 1, 3)$



In Exercises 51–54, show that $\partial^2 z/(\partial x \partial y) = \partial^2 z/(\partial y \partial x)$.

51. $z = x^2 - 2xy + 3y^2$

52. $z = x^4 - 3x^2y^2 + y^4$

53. $z = \frac{e^{2xy}}{4x}$

54. $z = \frac{x^2 - y^2}{2xy}$

In Exercises 55–62, find the second partial derivatives

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial y \partial x}, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y}.$$

55. $z = x^3 - 4y^2$

56. $z = 3x^2 - xy + 2y^3$

57. $z = 4x^3 + 3xy^2 - 4y^3$

58. $z = \sqrt{9 - x^2 - y^2}$

59. $z = \frac{xy}{x - y}$

60. $z = \frac{x}{x + y}$

61. $z = xe^{-y^2}$

62. $z = xe^y + ye^x$

In Exercises 63–66, evaluate the second partial derivatives f_{xx}, f_{xy}, f_{yy}, and f_{yx} at the point.

	Function	*Point*
63.	$f(x, y) = x^4 - 3x^2y^2 + y^2$	$(1, 0)$
64.	$f(x, y) = \sqrt{x^2 + y^2}$	$(0, 2)$
65.	$f(x, y) = \ln(x - y)$	$(2, 1)$
66.	$f(x, y) = x^2e^y$	$(-1, 0)$

67. ***Marginal Cost*** A company manufactures two models of bicycles: a mountain bike and a racing bike. The cost function for producing x mountain bikes and y racing bikes is given by

$$C = 10\sqrt{xy} + 149x + 189y + 675.$$

Find the marginal costs ($\partial C/\partial x$ and $\partial C/\partial y$) when $x = 120$ and $y = 160$.

68. ***Marginal Revenue*** A pharmaceutical corporation has two plants that produce the same over-the-counter medicine. If x_1 and x_2 are the numbers of units produced at plant 1 and plant 2, respectively, then the total revenue for the product is given by

$$R = 200x_1 + 200x_2 - 4x_1^2 - 8x_1x_2 - 4x_2^2.$$

If $x_1 = 4$ and $x_2 = 12$, find the following.

(a) The marginal revenue for plant 1, $\partial R/\partial x_1$

(b) The marginal revenue for plant 2, $\partial R/\partial x_2$

69. ***Marginal Pro***
the Cobb-Dougl
$f(x, y) = 100x^{0.}$
(a) Find the mai
(b) Find the mai

70. ***Marginal Pro***
production funct

71. ***Complementar***
notation of Exar
demands for pro
the prices of pr
whether the dem
tary or substitute
(a) $x_1 = 150 -$
(b) $x_1 = 150 -$
(c) $x_1 = \frac{1000}{\sqrt{p_1 p_2}}$

72. ***Psychology*** E
gence test called
known as the *IQ*
vidual's mental
chronological ag
The result is the
$IQ(M, C) =$
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73. ***Education*** Le
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such that $\partial N/\partial p$
interpret the fact

74. ***Chemistry*** Th
plate is given by
$T = 500 - 0.$
where x and y ar
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x- and y-axes.

75. ***Chemistry*** A r
two average pers
model for this in
$A = 0.885t -$
where A is the ap
ture, and h is t
Source: The UM

69. ***Marginal Productivity*** Let $x = 1000$ and $y = 500$ in the Cobb-Douglas production function given by

$f(x, y) = 100x^{0.6}y^{0.4}$.

(a) Find the marginal productivity of labor, $\partial f/\partial x$.

(b) Find the marginal productivity of capital, $\partial f/\partial y$.

70. ***Marginal Productivity*** Repeat Exercise 69 for the production function given by $f(x, y) = 100x^{0.75}y^{0.25}$.

71. ***Complementary and Substitute Products*** Using the notation of Example 4 in this section, let x_1 and x_2 be the demands for products 1 and 2, respectively, and p_1 and p_2 the prices of products 1 and 2, respectively. Determine whether the demand functions below describe complementary or substitute product relationships.

(a) $x_1 = 150 - 2p_1 - \frac{5}{2}p_2, \quad x_2 = 350 - \frac{3}{2}p_1 - 3p_2$

(b) $x_1 = 150 - 2p_1 + 1.8p_2, \quad x_2 = 350 + \frac{3}{4}p_1 - 1.9p_2$

(c) $x_1 = \dfrac{1000}{\sqrt{p_1p_2}}, \quad x_2 = \dfrac{750}{p_2\sqrt{p_1}}$

72. ***Psychology*** Early in the twentieth century, an intelligence test called the *Stanford-Binet Test* (more commonly known as the *IQ test*) was developed. In this test, an individual's mental age M is divided by the individual's chronological age C and the quotient is multiplied by 100. The result is the individual's IQ.

$$IQ(M, C) = \frac{M}{C} \times 100$$

Find the partial derivatives of IQ with respect to M and with respect to C. Evaluate the partial derivatives at the point (12, 10) and interpret the result. *(Source: Adapted from Bernstein/Clark-Stewart/Roy/Wickens,* Psychology, *Fourth Edition)*

73. ***Education*** Let N be the number of applicants to a university, p the charge for food and housing at the university, and t the tuition. Suppose that N is a function of p and t such that $\partial N/\partial p < 0$ and $\partial N/\partial t < 0$. How would you interpret the fact that both partials are negative?

74. ***Chemistry*** The temperature at any point (x, y) in a steel plate is given by

$$T = 500 - 0.6x^2 - 1.5y^2$$

where x and y are measured in meters. At the point (2, 3), find the rate of change of the temperature with respect to the distance moved along the plate in the directions of the x- and y-axes.

75. ***Chemistry*** A measure of what hot weather feels like to two average persons is the Apparent Temperature Index. A model for this index is

$$A = 0.885t - 78.7h + 1.20th + 2.70$$

where A is the apparent temperature, t is the air temperature, and h is the relative humidity in decimal form. *(Source: The UMAP Journal)*

(a) Find $\partial A/\partial t$ and $\partial A/\partial h$ when $t = 90$°F and $h = 0.80$.

(b) Which has a greater effect on A, air temperature or humidity? Explain your reasoning.

76. ***Marginal Utility*** The utility function $U = f(x, y)$ is a measure of the utility (or satisfaction) derived by a person from the consumption of two goods x and y. Suppose the utility function is given by

$U = -5x^2 + xy - 3y^2$.

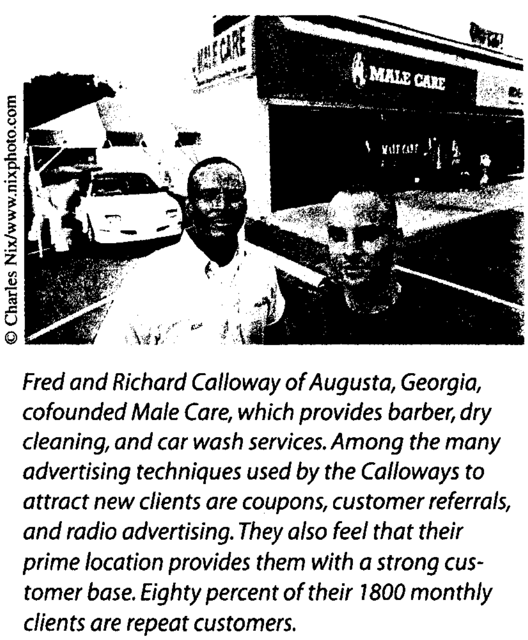
(a) Determine the marginal utility of good x.

(b) Determine the marginal utility of good y.

(c) When $x = 2$ and $y = 3$, should a person consume one more unit of good x or one more unit of good y? Explain your reasoning.

(d) Use a three-dimensional graphing utility to graph the function. Interpret the marginal utilities of goods x and y graphically.

BUSINESS CAPSULE



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Fred and Richard Calloway of Augusta, Georgia, cofounded Male Care, which provides barber, dry cleaning, and car wash services. Among the many advertising techniques used by the Calloways to attract new clients are coupons, customer referrals, and radio advertising. They also feel that their prime location provides them with a strong customer base. Eighty percent of their 1800 monthly clients are repeat customers.

77. ***Research Project*** Use your school's library, the Internet, or some other reference source to research a company that increased the demand for its product by creative advertising. Write a paper about the company. Use graphs to show how a change in demand is related to a change in the marginal utility of a product or service.

PREREQUISITE REVIEW 7.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–8, solve the system of equations.

1. $\begin{cases} 5x = 15 \\ 3x - 2y = 5 \end{cases}$

2. $\begin{cases} \frac{1}{2}y = 3 \\ -x + 5y = 19 \end{cases}$

3. $\begin{cases} x + y = 5 \\ x - y = -3 \end{cases}$

4. $\begin{cases} x + y = 8 \\ 2x - y = 4 \end{cases}$

5. $\begin{cases} 2x - y = 8 \\ 3x - 4y = 7 \end{cases}$

6. $\begin{cases} 2x - 4y = 14 \\ 3x + y = 7 \end{cases}$

7. $\begin{cases} x^2 + x = 0 \\ 2yx + y = 0 \end{cases}$

8. $\begin{cases} 3y^2 + 6y = 0 \\ xy + x + 2 = 0 \end{cases}$

In Exercises 9–14, find all first and second partial derivatives of the function.

9. $z = 4x^3 - 3y^2$
10. $z = 2x^5 - y^3$
11. $z = x^4 - \sqrt{xy} + 2y$
12. $z = 2x^2 - 3xy + y^2$
13. $z = ye^{xy^2}$
14. $z = xe^{xy}$

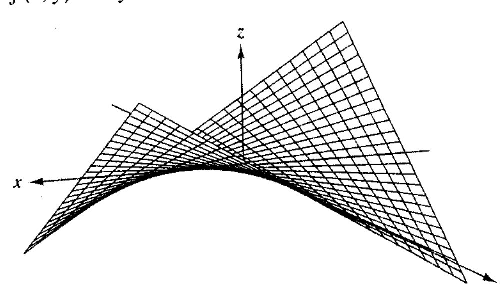
EXERCISES 7.5

In Exercises 1–4, find any critical points and relative extrema of the function.

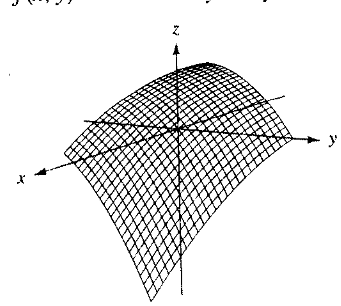
1. $f(x, y) = x^2 - y^2 + 4x - 8y - 11$
2. $f(x, y) = x^2 + y^2 + 2x - 6y + 6$
3. $f(x, y) = \sqrt{x^2 + y^2 + 1}$
4. $f(x, y) = \sqrt{25 - (x - 2)^2 - y^2}$

In Exercises 5–20, examine each function for relative extrema and saddle points.

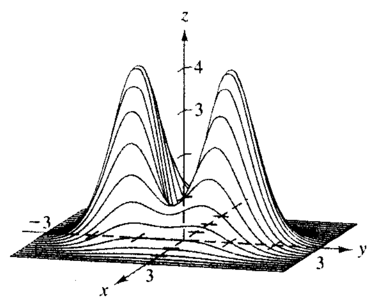
5. $f(x, y) = (x - 1)^2 + (y - 3)^2$
6. $f(x, y) = 9 - (x - 3)^2 - (y + 2)^2$
7. $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$
8. $f(x, y) = -x^2 - 5y^2 + 8x - 10y - 13$
9. $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$
10. $f(x, y) = x^2 + 6xy + 10y^2 - 4y + 4$
11. $f(x, y) = 3x^2 + 2y^2 - 12x - 4y + 7$
12. $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$
13. $f(x, y) = x^2 - y^2 + 4x - 4y - 8$
14. $f(x, y) = x^2 - 3xy - y^2$
15. $f(x, y) = xy$



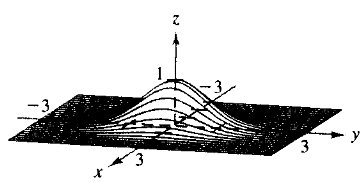
16. $f(x, y) = 12x + 12y - xy - x^2 - y^2$



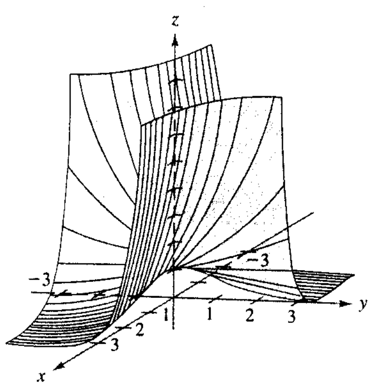
17. $f(x, y) = (x^2 + 4y^2)e^{1-x^2-y^2}$



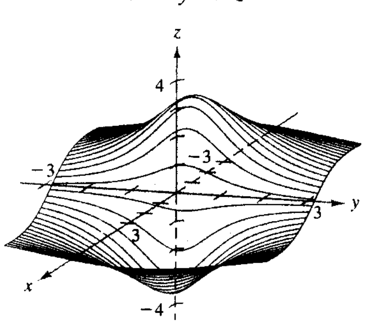
18. $f(x, y) = e^{-(x^2+y^2)}$



19. $f(x, y) = e^{xy}$



20. $f(x, y) = -\dfrac{4x}{x^2 + y^2 + 1}$



In Exercises 21–24, determine whether there is a relative maximum, a relative minimum, a saddle point, or insufficient information to determine the nature of the function $f(x, y)$ at the critical point (x_0, y_0).

21. $f_{xx}(x_0, y_0) = 16$
$f_{yy}(x_0, y_0) = 4$
$f_{xy}(x_0, y_0) = 8$

22. $f_{xx}(x_0, y_0) = -4$
$f_{yy}(x_0, y_0) = -6$
$f_{xy}(x_0, y_0) = 3$

23. $f_{xx}(x_0, y_0) = -7$
$f_{yy}(x_0, y_0) = 4$
$f_{xy}(x_0, y_0) = 9$

24. $f_{xx}(x_0, y_0) = 20$
$f_{yy}(x_0, y_0) = 8$
$f_{xy}(x_0, y_0) = 9$

In Exercises 25–30, find the critical points and test for relative extrema. List the critical points for which the Second-Partials Test fails.

25. $f(x, y) = (xy)^2$
26. $f(x, y) = \sqrt{x^2 + y^2}$
27. $f(x, y) = x^3 + y^3$
28. $f(x, y) = x^3 + y^3 - 3x^2 + 6y^2 + 3x + 12y + 7$
29. $f(x, y) = x^{2/3} + y^{2/3}$
30. $f(x, y) = (x^2 + y^2)^{2/3}$

In Exercises 31 and 32, find the critical points of the function and, from the form of the function, determine whether each critical point is a relative maximum or a relative minimum.

31. $f(x, y, z) = (x - 1)^2 + (y + 3)^2 + z^2$
32. $f(x, y, z) = 6 - [x(y + 2)(z - 1)]^2$

In Exercises 33–36, find three positive numbers x, y, and z that satisfy the given conditions.

33. The sum is 30 and the product is maximum.
34. The sum is 32 and $P = xy^2z$ is maximum.
35. The sum is 30 and the sum of the squares is minimum.
36. The sum is 1 and the sum of the squares is minimum.

37. ***Revenue*** A company manufactures two products. The total revenue from x_1 units of product 1 and x_2 units of product 2 is given by

$$R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2.$$

Find x_1 and x_2 so as to maximize the revenue.

38. ***Revenue*** A retail outlet sells two competitive products, the prices of which are p_1 and p_2. Find p_1 and p_2 so as to maximize the total revenue

$$R = 500p_1 + 800p_2 + 1.5p_1p_2 - 1.5p_1^2 - p_2^2.$$

Revenue In Exer
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39. $x_1 = 1000 - 2$
40. $x_1 = 1000 - 4$

41. ***Profit*** A cor
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$C_1 = 0.05x$

and

$C_2 = 0.03x$

respectively. Th
by

$p = 225 -$

and so the total

$R = [225 -$

Find the produ
maximize the p

$P = R - C$

42. ***Cost*** The ma
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43. ***Volume*** Find
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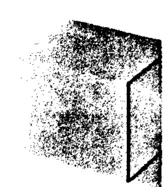

44. ***Volume*** Show
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45. ***Hardy-Weinb***
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wo competitive products,
. Find p_1 and p_2 so as to

$_2 - 1.5p_1^2 - p_2^2$.

Revenue In Exercises 39 and 40, find p_1 and p_2 so as to maximize the total revenue $R = x_1p_1 + x_2p_2$ for a retail outlet that sells two competitive products with the given demand functions.

39. $x_1 = 1000 - 2p_1 + p_2,\ x_2 = 1500 + 2p_1 - 1.5p_2$

40. $x_1 = 1000 - 4p_1 + 2p_2,\ x_2 = 900 + 4p_1 - 3p_2$

41. ***Profit*** A corporation manufactures a high-performance automobile engine product at two locations. The cost functions for producing x_1 units at location 1 and x_2 units at location 2 are given by

$$C_1 = 0.05x_1^2 + 15x_1 + 5400$$

and

$$C_2 = 0.03x_2^2 + 15x_2 + 6100$$

respectively. The demand function for the product is given by

$$p = 225 - 0.4(x_1 + x_2)$$

and so the total revenue function is given by

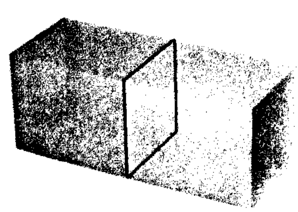
$$R = [225 - 0.4(x_1 + x_2)](x_1 + x_2).$$

Find the production levels at the two locations that will maximize the profit

$$P = R - C_1 - C_2.$$

42. ***Cost*** The material for constructing the base of an open box costs 1.5 times as much as the material for constructing the sides. Find the dimensions of the box of largest volume that can be made for a fixed amount of money C.

43. ***Volume*** Find the dimensions of a rectangular package of largest volume that may be sent by a shipping company assuming that the sum of the length and the girth (perimeter of a cross section) cannot exceed 144 inches.



44. ***Volume*** Show that a rectangular box of given volume and minimum surface area is a cube.

45. ***Hardy-Weinberg Law*** Common blood types are determined genetically by the three alleles A, B, and O. (An allele is any of a group of possible mutational forms of a gene.) A person whose blood type is AA, BB, or OO is homozygous. A person whose blood type is AB, AO, or BO is heterozygous. The Hardy-Weinberg Law states that the proportion P of heterozygous individuals in any given population is modeled by

$$P(p, q, r) = 2pq + 2pr + 2qr$$

where p represents the percent of allele A in the population, q represents the percent of allele B in the population, and r represents the percent of allele O in the population. Use the fact that $p + q + r = 1$ (the sum of the three must equal 100%) to show that the maximum proportion of heterozygous individuals in any population is $\frac{2}{3}$.

46. ***Biology*** A lake is to be stocked with smallmouth and largemouth bass. Let x represent the number of smallmouth bass and let y represent the number of largemouth bass in the lake. The weight of each fish is dependent on the population densities. After a six-month period, the weight of a single smallmouth bass is given by

$$W_1 = 3 - 0.002x - 0.005y$$

and the weight of a single largemouth bass is given by

$$W_2 = 4.5 - 0.003x - 0.004y.$$

Assuming that no fish die during the six-month period, how many smallmouth and largemouth bass should be stocked in the lake so that the *total* weight T of bass in the lake is a maximum?

47. ***Medicine*** In order to treat a certain bacterial infection, a combination of two drugs is being tested. Studies have shown that the duration of the infection in laboratory tests can be modeled by

$$D(x, y) = x^2 + 2y^2 - 18x - 24y + 2xy + 120$$

where x is the dose in hundreds of milligrams of the first drug and y is the dose in hundreds of milligrams of the second drug. Determine the partial derivatives of D with respect to x and with respect to y. Find the amount of each drug necessary to minimize the duration of the infection.

True or False? In Exercises 48–51, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

48. If $d > 0$ and $f_x(a, b) < 0$, then $f(a, b)$ is a relative minimum

49. A saddle point always occurs at a critical point.

50. If $f(x, y)$ has a relative maximum (x_0, y_0, z_0), then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

51. The function

$$f(x, y) = \sqrt[3]{x^2 + y^2}$$

has a relative maximum at the origin.

PREREQUISITE REVIEW 7.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–6, solve the system of linear equations.

1. $\begin{cases} 4x - 6y = 3 \\ 2x + 3y = 2 \end{cases}$

2. $\begin{cases} 6x - 6y = 5 \\ -3x - y = 1 \end{cases}$

3. $\begin{cases} 5x - y = 25 \\ x - 5y = 15 \end{cases}$

4. $\begin{cases} 4x - 9y = 5 \\ -x + 8y = -2 \end{cases}$

5. $\begin{cases} 2x - y + z = 3 \\ 2x + 2y + z = 4 \\ -x + 2y + 3z = -1 \end{cases}$

6. $\begin{cases} -x - 4y + 6z = -2 \\ x - 3y - 3z = 4 \\ 3x + y + 3z = 0 \end{cases}$

In Exercises 7–10, find all first partial derivatives.

7. $f(x, y) = x^2y + xy^2$

8. $f(x, y) = 25(xy + y^2)^2$

9. $f(x, y, z) = x(x^2 - 2xy + yz)$

10. $f(x, y, z) = z(xy + xz + yz)$

EXERCISES 7.6

In Exercises 1–12, use Lagrange multipliers to find the given extremum. In each case, assume that x and y are positive.

	Objective Function	*Constraint*
1.	Maximize $f(x, y) = xy$	$x + y = 10$
2.	Maximize $f(x, y) = xy$	$2x + y = 4$
3.	Minimize $f(x, y) = x^2 + y^2$	$x + y - 4 = 0$
4.	Minimize $f(x, y) = x^2 + y^2$	$-2x - 4y + 5 = 0$
5.	Maximize $f(x, y) = x^2 - y^2$	$y - x^2 = 0$
6.	Minimize $f(x, y) = x^2 - y^2$	$x - 2y + 6 = 0$
7.	Maximize $f(x, y) = 3x + xy + 3y$	$x + y = 25$
8.	Maximize $f(x, y) = 3x + y + 10$	$x^2y = 6$
9.	Maximize $f(x, y) = \sqrt{6 - x^2 - y^2}$	$x + y - 2 = 0$
10.	Minimize $f(x, y) = \sqrt{x^2 + y^2}$	$2x + 4y - 15 = 0$
11.	Maximize $f(x, y) = e^{xy}$	$x^2 + y^2 - 8 = 0$
12.	Minimize $f(x, y) = 2x + y$	$xy = 32$

In Exercises 13–18, use Lagrange multipliers to find the given extremum. In each case, assume that x, y, and z are positive.

13. Minimize $f(x, y, z) = 2x^2 + 3y^2 + 2z^2$

Constraint: $x + y + z - 24 = 0$

14. Maximize $f(x, y, z) = xyz$

Constraint: $x + y + z - 6 = 0$

15. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraint: $x + y + z = 1$

16. Minimize $f(x, y) = x^2 - 8x + y^2 - 12y + 48$

Constraint: $x + y = 8$

17. Maximize $f(x, y, z) = x + y + z$

Constraint: $x^2 + y^2 + z^2 = 1$

18. Maximize $f(x, y, z) = x^2y^2z^2$

Constraint: $x^2 + y^2 + z^2 = 1$

In Exercises 19 and 20, use Lagrange multipliers with the objective function

$$f(x, y, z, w) = 2x^2 + y^2 + z^2 + 2w^2$$

and with the given constraints to find the given extremum. In each case, assume that x, y, z, and w are nonnegative.

19. Maximize $f(x, y, z, w)$

Constraint: $2x + 2y + z + w = 2$

20. Maximize $f(x, y, z, w)$

Constraint: $x + y + 2z + 2w = 4$

In Exercises 21–24, use Lagrange multipliers to find the given extremum of f subject to two constraints. In each case, assume that x, y, and z are nonnegative.

21. Maximize $f(x, y, z) = xyz$

Constraints: $x + y + z = 24,\ x - y + z = 12$

22. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraints: $x + 2z = 4,\ x + y = 8$

23. Maximize $f(x, y, z) = xyz$

Constraints: $x^2 + z^2 = 5,\ x - 2y = 0$

24. Maximize $f(x, y, z) = xy + yz$

Constraints: $x + 2y = 6,\ x - 3z = 0$

In Exercises 25 and 26, use a spreadsheet to find the given extremum. In each case, assume that x, y, and z are nonnegative.

25. Maximize $f(x, y, z) = xyz$

Constraints: $x + 3y = 6,\ x - 2z = 0$

26. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraints: $x + 2y = 8,\ x + z = 4$

In Exercises 27–30, find three positive numbers x, y, and z that satisfy the given conditions.

27. The sum is 120 and the product is maximum.

28. The sum is 120 and the sum of the squares is minimum.

29. The sum is S and the product is maximum.

30. The sum is S and the sum of the squares is minimum.

In Exercises 31–34, find the minimum distance from the curve or surface to the given point. (*Hint:* Start by minimizing the square of the distance.)

31. Line: $x + 2y = 5,\ (0, 0)$

Minimize $d^2 = x^2 + y^2$

32. Circle: $(x - 4)^2 + y^2 = 4,\ (0, 10)$

Minimize $d^2 = x^2 + (y - 10)^2$

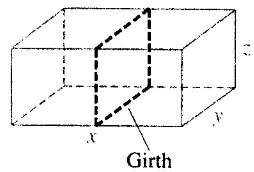
33. Plane: $x + y + z = 1,\ (2, 1, 1)$

Minimize $d^2 = (x - 2)^2 + (y - 1)^2 + (z - 1)^2$

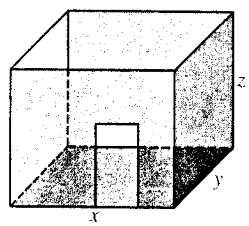
34. Cone: $z = \sqrt{x^2 + y^2},\ (4, 0, 0)$

Minimize $d^2 = (x - 4)^2 + y^2 + z^2$

35. ***Volume*** Find the dimensions of the rectangular package of largest volume subject to the constraint that the sum of the length and the girth cannot exceed 108 inches (see figure). (*Hint:* Maximize $V = xyz$ subject to the constraint $x + 2y + 2z = 108$.)



36. ***Cost*** In redecorating an office, the cost for new carpeting is five times the cost of wallpapering a wall. Find the dimensions of the largest office that can be redecorated for a fixed cost C (see figure). (*Hint:* Maximize $V = xyz$ subject to $5xy + 2xz + 2yz = C$.)



37. ***Cost*** A cargo container (in the shape of a rectangular solid) must have a volume of 480 cubic feet. Use Lagrange multipliers to find the dimensions of the container of this size that has a minimum cost, if the bottom will cost \$5 per square foot to construct and the sides and top will cost \$3 per square foot to construct.

38. ***Cost*** A manufacturer has an order for 1000 units of fine paper that can be produced at two locations. Let x_1 and x_2 be the numbers of units produced at the two plants. Find the number of units that should be produced at each plant to minimize the cost if the cost function is given by

$$C = 0.25x_1^2 + 25x_1 + 0.05x_2^2 + 12x_2.$$

39. ***Cost*** A manufacturer has an order for 2000 units of all-terrain vehicle tires that can be produced at two locations. Let x_1 and x_2 be the numbers of units produced at the two plants. The cost function is modeled by

$$C = 0.25x_1^2 + 10x_1 + 0.15x_2^2 + 12x_2.$$

Find the number of units that should be produced at each plant to minimize the cost.

40. ***Hardy-Weinberg Law*** Repeat Exercise 45 in Section 7.5 using Lagrange multipliers—that is, maximize

$$P(p, q, r) = 2pq + 2pr + 2qr$$

subject to the constraint

$$p + q + r = 1.$$

41. ***Least-Cost R***
pany is given b
$f(x, y) = 10$
where x is the n
of units of capi
capital costs \$3
tion goal of 20,(
(a) Find the nu
to meet the
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Marginal produ
Marginal produ
This proportio
Equimarginal R

42. ***Least-Cost Ru***
function given b
$f(x, y) = 10$

43. ***Production*** T
given by
$f(x, y) = 10$
where x is the nu
of units of capita
capital costs \$36
is limited to \$10
(a) Find the ma
turer.
(b) Find the ma
(c) Use the ma
maximum n
\$125,000 is

44. ***Production*** I
function given b
$f(x, y) = 10($

45. ***Biology*** A mi
um in which to g
of salt contained
$S = 12xyz$
where x, y, and
the medium. For
13% salt. Nutrie
per liter, respecti
should be used tc

46. ***Biology*** Repe
$S = 0.01x^2y^2$

z^2

the rectangular package
constraint that the sum
not exceed 108 inches
$V = xyz$ subject to the

e cost for new carpeting
ering a wall. Find the
hat can be redecorated
lint: Maximize $V = xyz$

shape of a rectangular
ubic feet. Use Lagrange
of the container of this
bottom will cost \$5 per
les and top will cost \$3

r for 1000 units of fine
locations. Let x_1 and x_2
the two plants. Find the
duced at each plant to
n is given by

$+ 12x_2.$

der for 2000 units of
produced at two loca-
of units produced at the
deled by

$+ 12x_2.$

ld be produced at each

Exercise 45 in Section
it is, maximize

41. ***Least-Cost Rule*** The production function for a company is given by

$$f(x, y) = 100x^{0.25}y^{0.75}$$

where x is the number of units of labor and y is the number of units of capital. Suppose that labor costs \$48 per unit, capital costs \$36 per unit, and management sets a production goal of 20,000 units.

(a) Find the numbers of units of labor and capital needed to meet the production goal while minimizing the cost.

(b) Show that the conditions of part (a) are met when

$$\frac{\text{Marginal productivity of labor}}{\text{Marginal productivity of capital}} = \frac{\text{unit price of labor}}{\text{unit price of capital}}.$$

This proportion is called the *Least-Cost Rule* (or Equimarginal Rule).

42. ***Least-Cost Rule*** Repeat Exercise 41 for the production function given by

$$f(x, y) = 100x^{0.6}y^{0.4}.$$

43. ***Production*** The production function for a company is given by

$$f(x, y) = 100x^{0.25}y^{0.75}$$

where x is the number of units of labor and y is the number of units of capital. Suppose that labor costs \$48 per unit and capital costs \$36 per unit. The total cost of labor and capital is limited to \$100,000.

(a) Find the maximum production level for this manufacturer.

(b) Find the marginal productivity of money.

(c) Use the marginal productivity of money to find the maximum number of units that can be produced if \$125,000 is available for labor and capital.

44. ***Production*** Repeat Exercise 43 for the production function given by

$$f(x, y) = 100x^{0.06}y^{0.04}.$$

45. ***Biology*** A microbiologist must prepare a culture medium in which to grow a certain type of bacteria. The percent of salt contained in this medium is given by

$$S = 12xyz$$

where x, y, and z are the nutrient solutions to be mixed in the medium. For the bacteria to grow, the medium must be 13% salt. Nutrient solutions x, y, and z cost \$1, \$2, and \$3 per liter, respectively. How much of each nutrient solution should be used to minimize the cost of the culture medium?

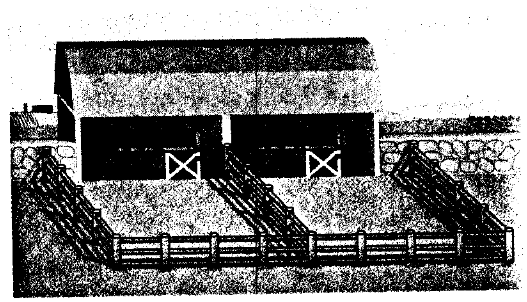
46. ***Biology*** Repeat Exercise 45 for a salt-content model of

$$S = 0.01x^2y^2z^2.$$

47. ***Construction*** A rancher plans to use an existing stone wall and the side of a barn as a boundary for two adjacent rectangular corrals. Fencing for the perimeter costs \$10 per foot. To separate the corrals, a fence that costs \$4 per foot will divide the region. The total area of the two corrals is to be 6000 square feet.

(a) Use Lagrange multipliers to find the dimensions that will minimize the cost of the fencing.

(b) What is the minimum cost?



48. ***Area*** Use Lagrange multipliers to show that the maximum area of a rectangle with dimensions x and y and a given perimeter P is $\frac{1}{16}P^2$.

49. ***Investment Strategy*** An investor is considering three different stocks in which to invest \$300,000. The average annual dividends for the stocks are

General Mills (G)	2.5%
Pepsico, Inc. (P)	1.4%
Sara Lee (S)	3.1%.

The amount invested in Pepsico, Inc. must follow the equation

$$3000(G) - 3000(S) + P^2 = 0.$$

How much should be invested in each stock to yield a maximum of dividends?

50. ***Investment Strategy*** An investor is considering three different stocks in which to invest \$20,000. The average annual dividends for the stocks are

General Motors (G)	5.2%
Campbell Soup (C)	2.7%
Kellogg Co. (K)	3.2%.

The amount invested in Campbell Soup must follow the equation

$$1000(K) - 1000(G) + C^2 = 0.$$

How much should be invested in each stock to yield a maximum of dividends?

51. ***Research Project*** Use your school's library, the Internet, or some other reference source to write a paper about two different types of available investment options. Find examples of each type and find the data about their dividends for the past 10 years. What are the similarities and differences between the two types?

PREREQUISITE REVIEW 7.8

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–12, evaluate the definite integral.

1. $\int_0^1 dx$
2. $\int_0^2 3\,dy$
3. $\int_1^4 2x^2\,dx$
4. $\int_0^1 2x^3\,dx$
5. $\int_1^2 (x^3 - 2x + 4)\,dx$
6. $\int_0^2 (4 - y^2)\,dy$
7. $\int_1^2 \frac{2}{7x^2}\,dx$
8. $\int_1^4 \frac{2}{\sqrt{x}}\,dx$
9. $\int_0^2 \frac{2x}{x^2 + 1}\,dx$
10. $\int_2^e \frac{1}{y - 1}\,dy$
11. $\int_0^2 xe^{x^2+1}\,dx$
12. $\int_0^1 e^{-2y}\,dy$

In Exercises 13–16, sketch the region bounded by the graphs of the equations.

13. $y = x,\ y = 0,\ x = 3$
14. $y = x,\ y = 3,\ x = 0$
15. $y = 4 - x^2,\ y = 0,\ x = 0$
16. $y = x^2,\ y = 4x$

EXERCISES 7.8

In Exercises 1–10, evaluate the partial integral.

1. $\int_0^x (2x - y)\,dy$
2. $\int_x^{x^2} \frac{y}{x}\,dy$
3. $\int_1^{2y} \frac{y}{x}\,dx$
4. $\int_0^{e^y} y\,dx$
5. $\int_0^{\sqrt{9-x^2}} x^2y\,dy$
6. $\int_{x^2}^{\sqrt{x}} (x^2 + y^2)\,dy$
7. $\int_{e^y}^{y} \frac{y \ln x}{x}\,dx$
8. $\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2)\,dx$
9. $\int_0^{x^3} ye^{-y/x}\,dy$
10. $\int_y^3 \frac{xy}{\sqrt{x^2 + 1}}\,dx$

In Exercises 11–24, evaluate the double integral.

11. $\int_0^2 \int_0^1 (x - y)\,dy\,dx$
12. $\int_0^2 \int_0^2 (6 - x^2)\,dy\,dx$
13. $\int_0^4 \int_0^3 xy\,dy\,dx$
14. $\int_0^1 \int_0^x \sqrt{1 - x^2}\,dy\,dx$
15. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x + y)\,dx\,dy$
16. $\int_0^2 \int_{3y^2-6y}^{2y-y^2} 3y\,dx\,dy$
17. $\int_1^2 \int_0^4 (x^2 - 2y^2 + 1)\,dx\,dy$
18. $\int_0^1 \int_y^{2y} (1 + 2x^2 + 2y^2)\,dx\,dy$
19. $\int_0^2 \int_0^{\sqrt{1-y^2}} -5xy\,dx\,dy$
20. $\int_0^4 \int_0^x \frac{2}{(x + 1)(y + 1)}\,dy\,dx$
21. $\int_0^2 \int_0^{4-x^2} x^3\,dy\,dx$
22. $\int_0^a \int_0^{a-x} (x^2 + y^2)\,dy\,dx$
23. $\int_0^\infty \int_0^\infty e^{-(x+y)/2}\,dy\,dx$
24. $\int_0^\infty \int_0^\infty xye^{-(x^2+y^2)}\,dx\,dy$

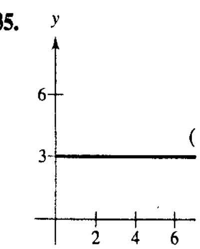
In Exercises 25–32, ske
double integral. Then
that both orders yield

25. $\int_0^1 \int_0^2 dy\,dx$
26. $\int_1^2 \int_2^4 dx\,dy$
27. $\int_0^1 \int_{2y}^2 dx\,dy$
28. $\int_0^4 \int_0^{\sqrt{x}} dy\,dx$
29. $\int_0^2 \int_{x/2}^1 dy\,dx$
30. $\int_0^4 \int_{\sqrt{x}}^2 dx\,dy$
31. $\int_0^1 \int_{y^2}^{\sqrt[3]{y}} dx\,dy$
32. $\int_{-2}^2 \int_0^{4-y^2} dx\,dy$

In Exercises 33 and 34
is necessary to change

33. $\int_0^3 \int_y^3 e^{x^2}\,dx\,dy$
34. $\int_0^2 \int_x^2 e^{-y^2}\,dy\,dx$

In Exercises 35–40, use
specified region.

35.


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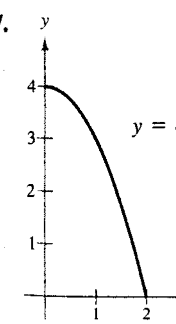


sections. You will

In Exercises 25–32, sketch the region R whose area is given by the double integral. Then change the order of integration and show that both orders yield the same area.

25. $\displaystyle\int_0^1\int_0^2 dy\,dx$

26. $\displaystyle\int_1^2\int_2^4 dx\,dy$

27. $\displaystyle\int_0^1\int_{2y}^2 dx\,dy$

28. $\displaystyle\int_0^4\int_0^{\sqrt{x}} dy\,dx$

29. $\displaystyle\int_0^2\int_{x/2}^1 dy\,dx$

30. $\displaystyle\int_0^4\int_{\sqrt{x}}^2 dx\,dy$

31. $\displaystyle\int_0^1\int_{y^2}^{\sqrt[3]{y}} dx\,dy$

32. $\displaystyle\int_{-2}^2\int_0^{4-y^2} dx\,dy$

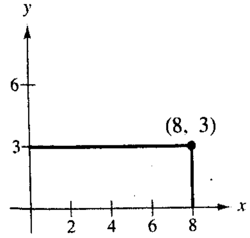
In Exercises 33 and 34, evaluate the double integral. Note that it is necessary to change the order of integration.

33. $\displaystyle\int_0^3\int_y^3 e^{x^2}\,dx\,dy$

34. $\displaystyle\int_0^2\int_x^2 e^{-y^2}\,dy\,dx$

In Exercises 35–40, use a double integral to find the area of the specified region.

35.



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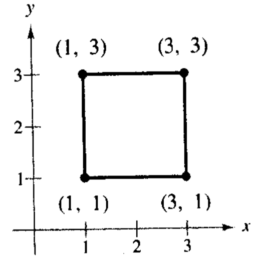

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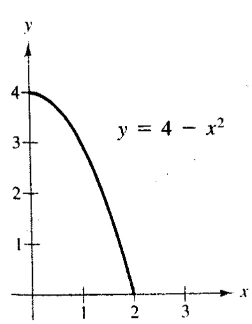

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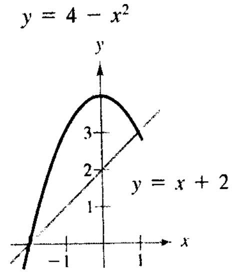

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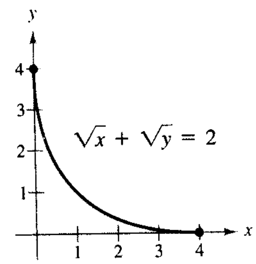

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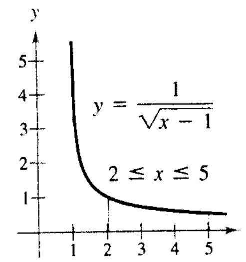


In Exercises 41–46, use a double integral to find the area of the region bounded by the graphs of the equations.

41. $y = 25 - x^2,\ y = 0$

42. $y = x^{3/2},\ y = x$

43. $5x - 2y = 0,\ x + y = 3,\ y = 0$

44. $xy = 9,\ y = x,\ y = 0,\ x = 9$

45. $y = x,\ y = 2x,\ x = 2$

46. $y = x^2 + 2x + 1,\ y = 3(x + 1)$

In Exercises 47–54, use a symbolic integration utility to evaluate the double integral.

47. $\displaystyle\int_0^1\int_0^2 e^{-x^2-y^2}\,dx\,dy$

48. $\displaystyle\int_0^2\int_{x^2}^{2x} (x^3 + 3y^2)\,dy\,dx$

49. $\displaystyle\int_1^2\int_0^x e^{xy}\,dy\,dx$

50. $\displaystyle\int_1^2\int_y^{2y} \ln(x + y)\,dx\,dy$

51. $\displaystyle\int_0^1\int_x^1 \sqrt{1 - x^2}\,dy\,dx$

52. $\displaystyle\int_0^3\int_0^{x^2} \sqrt{x}\sqrt{1 + x}\,dy\,dx$

53. $\displaystyle\int_0^2\int_{\sqrt{4-x^2}}^{4-x^2/4} \frac{xy}{x^2 + y^2 + 1}\,dy\,dx$

54. $\displaystyle\int_0^4\int_0^y \frac{2}{(x + 1)(y + 1)}\,dx\,dy$

True or False? In Exercises 55 and 56, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

55. Changing the order of integration will sometimes change the value of a double integral.

56. $\displaystyle\int_2^5\int_1^6 x\,dy\,dx = \int_1^6\int_2^5 x\,dx\,dy$

PREREQUISITE REVIEW 7.9

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, sketch the region that is described.

1. $0 \le x \le 2,\ 0 \le y \le 1$

2. $1 \le x \le 3,\ 2 \le y \le 3$

3. $0 \le x \le 4,\ 0 \le y \le 2x - 1$

4. $0 \le x \le 2,\ 0 \le y \le x^2$

In Exercises 5–10, evaluate the double integral.

5. $\displaystyle\int_0^1\int_1^2 dy\,dx$

6. $\displaystyle\int_0^3\int_1^3 dx\,dy$

7. $\displaystyle\int_0^1\int_0^x x\,dy\,dx$

8. $\displaystyle\int_0^4\int_1^y y\,dx\,dy$

9. $\displaystyle\int_1^3\int_x^{x^2} 2\,dy\,dx$

10. $\displaystyle\int_0^1\int_x^{-x^2+2} dy\,dx$

EXERCISES 7.9

In Exercises 1–8, sketch the region of integration and evaluate the double integral.

1. $\displaystyle\int_0^2\int_0^1 (3x + 4y)\,dy\,dx$

2. $\displaystyle\int_0^3\int_0^1 (2x + 6y)\,dy\,dx$

3. $\displaystyle\int_0^1\int_y^{\sqrt{y}} x^2y^2\,dx\,dy$

4. $\displaystyle\int_0^6\int_{y/2}^3 (x + y)\,dx\,dy$

5. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} y\,dy\,dx$

6. $\displaystyle\int_0^2\int_0^{4-x^2} xy^2\,dy\,dx$

7. $\displaystyle\int_{-a}^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$

8. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}} dy\,dx$

In Exercises 9–12, set up the integral for both orders of integration and use the more convenient order to evaluate the integral over the region R.

9. $\displaystyle\int_R\int xy\,dA$

R: rectangle with vertices at $(0, 0)$, $(0, 5)$, $(3, 5)$, $(3, 0)$

10. $\displaystyle\int_R\int x\,dA$

R: semicircle bounded by $y = \sqrt{25 - x^2}$ and $y = 0$

11. $\displaystyle\int_R\int \frac{y}{x^2 + y^2}\,dA$

R: triangle bounded by $y = x$, $y = 2x$, $x = 2$

12. $\displaystyle\int_R\int \frac{y}{1 + x^2}\,dA$

R: region bounded by $y = 0$, $y = \sqrt{x}$, $x = 4$

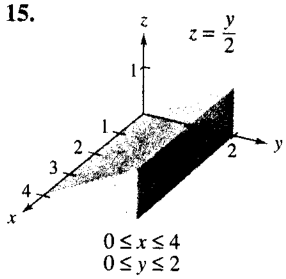
In Exercises 13 and 14, evaluate the double integral. Note that it is necessary to change the order of integration.

13. $\displaystyle\int_0^1\int_{y/2}^{1/2} e^{-x^2}\,dx\,dy$

14. $\displaystyle\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\,dy\,dx$

In Exercises 15–26, use a double integral to find the volume of the specified solid.

15.



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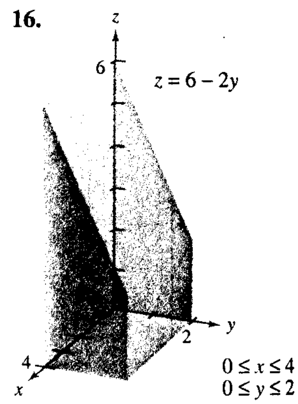

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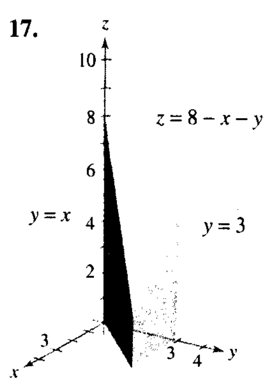

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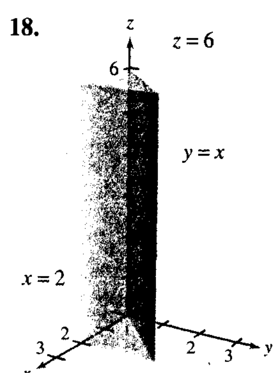

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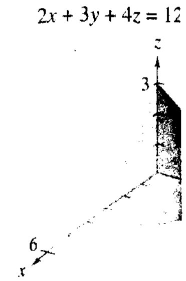

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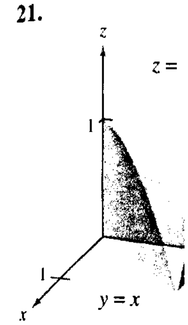

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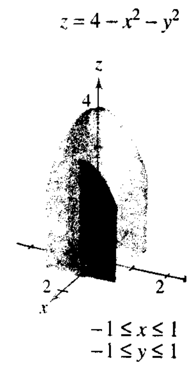

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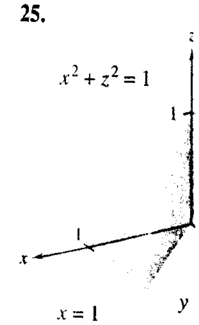


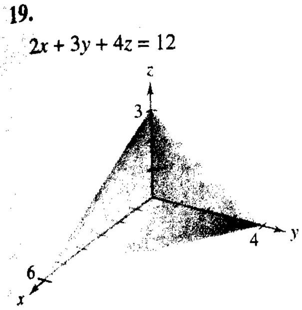
In Exercises 27–30, u

solid bounded by th

27. $z = xy,\ z = 0,$

28. $z = x,\ z = 0,$

19.



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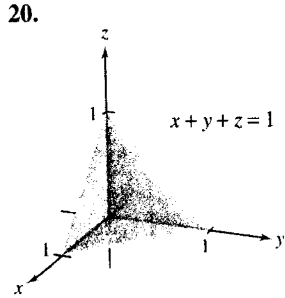

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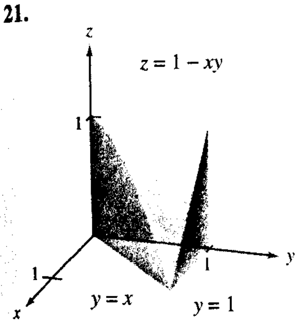

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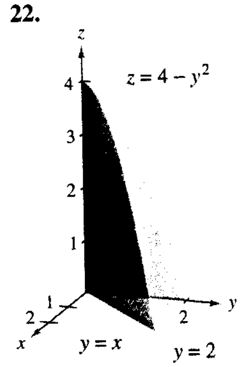

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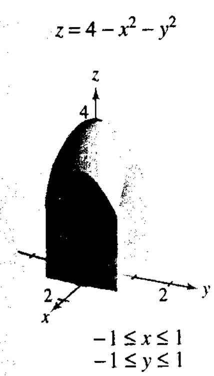

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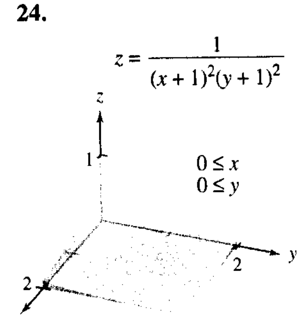

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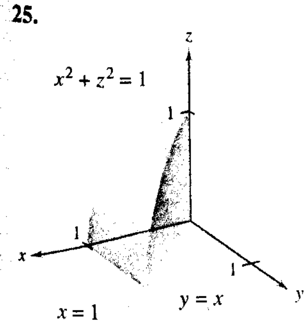

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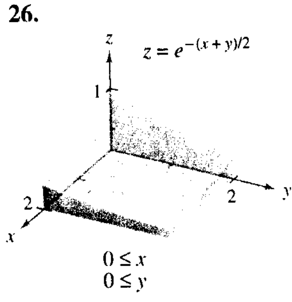

In Exercises 27–30, use a double integral to find the volume of the solid bounded by the graphs of the equations.

27. $z = xy,\ z = 0,\ y = 0,\ y = 4,\ x = 0,\ x = 1$

28. $z = x,\ z = 0,\ y = x,\ y = 0,\ x = 0,\ x = 4$

29. $z = x^2,\ z = 0,\ x = 0,\ x = 2,\ y = 0,\ y = 4$

30. $z = x + y,\ x^2 + y^2 = 4$ (first octant)

31. ***Population Density*** The population density (in people per square mile) for a coastal town can be modeled by

$$f(x, y) = \frac{120{,}000}{(2 + x + y)^3}$$

where x and y are measured in miles. What is the population inside the rectangular area defined by the vertices $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(2, 2)$?

32. ***Population Density*** The population density (in people per square mile) for a coastal town on an island can be modeled by

$$f(x, y) = \frac{5000xe^y}{1 + 2x^2}$$

where x and y are measured in miles. What is the population inside the rectangular area defined by the vertices $(0, 0)$, $(4, 0)$, $(0, -2)$, and $(4, -2)$?

In Exercises 33–36, find the average value of $f(x, y)$ over the region R.

	Integral	*Region R*
33.	$f(x, y) = x$	Rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 2)$, $(0, 2)$
34.	$f(x, y) = xy$	Rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 2)$, $(0, 2)$
35.	$f(x, y) = x^2 + y^2$	Square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$
36.	$f(x, y) = e^{x+y}$	Triangle with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$

37. ***Average Revenue*** A company sells two products whose demand functions are given by

$$x_1 = 500 - 3p_1 \quad \text{and} \quad x_2 = 750 - 2.4p_2.$$

So, the total revenue is given by

$$R = x_1p_1 + x_2p_2.$$

Estimate the average revenue if the price p_1 varies between \$50 and \$75 and the price p_2 varies between \$100 and \$150.

38. ***Average Weekly Profit*** A firm's weekly profit in marketing two products is given by

$$P = 192x_1 + 576x_2 - x_1^2 - 5x_2^2 - 2x_1x_2 - 5000$$

where x_1 and x_2 represent the numbers of units of each product sold weekly. Estimate the average weekly profit if x_1 varies between 40 and 50 units and x_2 varies between 45 and 50 units.