The term $b^2 - 4ac$ under the square root sign in the quadratic formula is called the **discriminant.** If the discriminant is nonnegative, the two solutions of the corresponding quadratic equation are real. (When the discriminant is equal to 0, the two solutions are identical.) If the discriminant is negative, the two solutions are complex conjugates of each other.

EXAMPLE 15 Without solving

$$2x^2 - 3x + 7 = 0$$

what can you say about the solution?

Solution We compute the discriminant

$$b^2 - 4ac = (-3)^2 - (4)(2)(7) = 9 - 56 = -47 < 0$$

Since the discriminant is negative, the equation $2x^2 - 3x + 7 = 0$ has two complex solutions, which are conjugates of each other. ■

Section 1.1 Problems

■ 1.1.1

1. Find the two numbers that have distance 3 from −1 by **(a)** measuring the distances on the real-number line and **(b)** solving an appropriate equation involving an absolute value.
2. Find all pairwise distances between the numbers −5, 2, and 7 by **(a)** measuring the distances on the real-number line and **(b)** computing the distances by using absolute values.
3. Solve the following equations:
(a) $|2x - 4| = 6$ **(b)** $|x - 3| = 2$
(c) $|2x + 3| = 5$ **(d)** $|7 - 3x| = -2$
4. Solve the following equations:
(a) $|2x + 4| = |5x - 2|$ **(b)** $|5 - 3u| = |3 + 2u|$
(c) $|4 + \frac{t}{2}| = |\frac{3}{2}t - 2|$ **(d)** $|2s - 3| = |7 - s|$
5. Solve the following inequalities:
(a) $|5x - 2| \le 4$ **(b)** $|1 - 3x| > 8$
(c) $|7x + 4| \ge 3$ **(d)** $|6 - 5x| < 7$
6. Solve the following inequalities:
(a) $|2x + 3| < 6$ **(b)** $|3 - 4x| \ge 2$
(c) $|x + 5| \le 1$ **(d)** $|7 - 2x| < 0$

■ 1.1.2

In Problems 7–42, determine the equation of the line that satisfies the stated requirements. Put the equation in standard form.

7. The line passing through (2, 4) with slope $-\frac{1}{3}$
8. The line passing through (1, −2) with slope 2
9. The line passing through (0, −2) with slope −3
10. The line passing through (−3, 5) with slope 1/2
11. The line passing through (−2, −3) and (1, 4)
12. The line passing through (−1, 4) and $(2, -\frac{1}{2})$
13. The line passing through (0, 4) and (3, 0)
14. The line passing through (1, −1) and (4, 5)
15. The horizontal line through $(3, \frac{3}{2})$
16. The horizontal line through (0, −1)
17. The vertical line through $(-1, \frac{7}{2})$
18. The vertical line through (2, −3)
19. The line with slope 3 and y-intercept (0, 2)
20. The line with slope −1 and y-intercept (0, −3)
21. The line with slope 1/2 and y-intercept (0, 2)
22. The line with slope −1/3 and y-intercept (0, −1)
23. The line with slope −2 and x-intercept (1, 0)
24. The line with slope 1 and x-intercept (−2, 0)
25. The line with slope −1/4 and x-intercept (3, 0)
26. The line with slope 1/5 and x-intercept (−1/2, 0)
27. The line passing through (2, −3) and parallel to
$$x + 2y - 4 = 0$$
28. The line passing through (1, 2) and parallel to
$$x - 3y - 6 = 0$$
29. The line passing through (−1, −1) and parallel to the line passing through (0, 1) and (3, 0)
30. The line passing through (2, −3) and parallel to the line passing through (0, −1) and (2, 1)
31. The line passing through (1, 4) and perpendicular to
$$2y - 5x + 7 = 0$$
32. The line passing through (−1, −1) and perpendicular to
$$x - y + 3 = 0$$
33. The line passing through (5, −1) and perpendicular to the line passing through (−2, 1) and (1, −2)
34. The line passing through (4, −1) and perpendicular to the line passing through (−2, 0) and (1, 1)
35. The line passing through (4, 2) and parallel to the horizontal line passing through (1, −2)
36. The line passing through (−1, 5) and parallel to the horizontal line passing through (2, −1)
37. The line passing through (−1, 1) and parallel to the vertical line passing through (2, −1)
38. The line passing through (3, 1) and parallel to the vertical line passing through (−1, −2)
39. The line passing through (1, −3) and perpendicular to the horizontal line passing through (−1, −1)

40. The line passing through (4, 2) and perpendicular to the horizontal line passing through (3, 1)

41. The line passing through (7, 3) and perpendicular to the vertical line passing through (−2, 4)

42. The line passing through (−2, 5) and perpendicular to the vertical line passing through (1, 4)

43. To convert a length measured in feet to a length measured in centimeters, we use the facts that a length measured in feet is proportional to a length measured in centimeters and that 1 ft corresponds to 30.5 cm. If x denotes the length measured in ft and y denotes the length measured in cm, then

$$y = 30.5x$$

(a) Explain how to use this relationship.

(b) Use the relationship to convert the following measurements into centimeters:

(i) 6 ft **(ii)** 3 ft, 2 in **(iii)** 1 ft, 7 in

(c) Use the relationship to convert the following measurements into ft:

(i) 173 cm **(ii)** 75 cm **(iii)** 48 cm

44. **(a)** To convert the weight of an object from kilograms (kg) to pounds (lb), you use the facts that a weight measured in kilograms is proportional to a weight measured in pounds and that 1 kg corresponds to 2.20 lb. Find an equation that relates weight measured in kilograms to weight measured in pounds.

(b) Use your answer in (a) to convert the following measurements:

(i) 63 lb **(ii)** 150 lb **(iii)** 2.5 kg **(iv)** 140 kg

45. Assume that the distance a car travels is proportional to the time it takes to cover the distance. Find an equation that relates distance and time if it takes the car 15 min to travel 10 mi. What is the constant of proportionality if distance is measured in miles and time is measured in hours?

46. Assume that the number of seeds a plant produces is proportional to its aboveground biomass. Find an equation that relates number of seeds and aboveground biomass if a plant that weighs 217 g has 17 seeds.

47. Experimental study plots are often squares of length 1 m. If 1 ft corresponds to 0.305 m, compute the area of a square plot of length 1 m in ft^2.

48. Large areas are often measured in hectares (ha) or in acres. If 1 ha = 10,000 m^2 and 1 acre = 4046.86 m^2, how many acres is 1 hectare?

49. To convert the volume of a liquid measured in ounces to a volume measured in liters, we use the fact that 1 liter equals 33.81 ounces. Denote by x the volume measured in ounces and by y the volume measured in liters. Assume a linear relationship between these two units of measurements.

(a) Find the equation relating x and y.

(b) A typical soda can contains 12 ounces of liquid. How many liters is this?

50. To convert a distance measured in miles to a distance measured in kilometers, we use the fact that 1 mile equals 1.609 kilometers. Denote by x the distance measured in miles and by y the distance measured in kilometers. Assume a linear relationship between these two units of measurements.

(a) Find an equation relating x and y.

(b) The distance between Minneapolis and Madison is 261 miles. How many kilometers is this?

51. Car speed in many countries is measured in kilometers per hour. In the United States, car speed is measured in miles per hour. To convert between these units, use the fact that 1 mile equals 1.609 kilometers.

(a) The speed limit on many U.S. highways is 55 miles per hour. Convert this number into kilometers per hour.

(b) The recommended speed limit on German highways is 130 kilometers per hour. Convert this number into miles per hour.

To measure temperature, three scales are commonly used: Fahrenheit, Celsius, and Kelvin. These scales are linearly related. We discuss these scales in Problems 52 and 53.

52. **(a)** The Celsius scale is devised so that 0°C is the freezing point of water (at 1 atmosphere of pressure) and 100°C is the boiling point of water (at 1 atmosphere of pressure). If you are more familiar with the Fahrenheit scale, then you know that water freezes at 32°F and boils at 212°F. Find a linear equation that relates temperature measured in degrees Celsius and temperature measured in degrees Fahrenheit.

(b) The normal body temperature in humans ranges from 97.6°F to 99.6°F. Convert this temperature range into degrees Celsius.

53. **(a)** The Kelvin (K) scale is an **absolute** scale of temperature. The zero point of the scale (0 K) denotes **absolute zero**, the coldest possible temperature; that is, no body can have a temperature below 0 K. It has been determined experimentally that 0 K corresponds to −273.15°C. If 1 K denotes the same temperature difference as 1°C, find an equation that relates the Kelvin and Celsius scales.

(b) Pure nitrogen and pure oxygen can be produced cheaply by first liquefying purified air and then allowing the temperature of the liquid air to rise slowly. Since nitrogen and oxygen have different boiling points, they are distilled at different temperatures. The boiling point of nitrogen is 77.4 K and of oxygen is 90.2 K. Convert each of these boiling-point temperatures into Celsius. If you solved Problem 52(a), convert the boiling-point temperatures into Fahrenheit as well. Consider the two techniques described for distilling nitrogen and oxygen. Which element gets distilled first?

54. Use the following steps to show that if two nonvertical lines l_1 and l_2 with slopes m_1 and m_2, respectively, are perpendicular, then $m_1 m_2 = -1$: Assume that $m_1 < 0$ and $m_2 > 0$.

(a) Use a graph to show that if θ_1 and θ_2 are the respective angles of inclination of the lines l_1 and l_2, then $\theta_1 = \theta_2 + \frac{\pi}{2}$. (The angle of inclination of a line is the angle $\theta \in [0, \pi)$ between the line and the positively directed x-axis.)

(b) Use the fact that $\tan(\pi - x) = -\tan x$ to show that $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.

(c) Use the fact that $\tan(\frac{\pi}{2} - x) = \cot x$ and $\cot(-x) = -\cot x$ to show that $m_1 = -\cot\theta_2$.

(d) From the latter equation, deduce the truth of the claim set forth at the beginning of this problem.

■ 1.1.3

55. Find the equation of a circle with center (−1, 4) and radius 3.

56. Find the equation of a circle with center (2, 3) and radius 4.

57. **(a)** Find the equation of a circle with center (2, 5) and radius 3.

(b) Where does the circle intersect the y-axis?

(c) Does the circle intersect the x-axis? Explain.

58. **(a)** Find all possible radii of a circle centered at (3, 6) so that the circle intersects only one axis.

(b) Find all possible radii of a circle centered at (3, 6) so that the circle intersects both axes.

59. Find the center and the radius of the circle given by the equation

$$(x-2)^2+y^2=16$$

60. Find the center and the radius of the circle given by the equation

$$(x+1)^2+(y-3)^2=9$$

61. Find the center and the radius of the circle given by the equation

$$0=x^2+y^2-4x+2y-11$$

(To do this, you must complete the squares.)

62. Find the center and the radius of the circle given by the equation

$$x^2+y^2+2x-4y+1=0$$

(To do this, you must complete the squares.)

■ 1.1.4

63. (a) Convert 75° to radian measure.
(b) Convert $\frac{17}{12}\pi$ to degree measure.

64. (a) Convert −15° to radian measure.
(b) Convert $\frac{3}{4}\pi$ to degree measure.

65. Evaluate the following expressions without using a calculator:
(a) $\sin(-\frac{5\pi}{4})$ (b) $\cos(\frac{5\pi}{6})$ (c) $\tan(\frac{\pi}{3})$

66. Evaluate the following expressions without using a calculator:
(a) $\sin(\frac{3\pi}{4})$ (b) $\cos(-\frac{13\pi}{6})$ (c) $\tan(\frac{4\pi}{3})$

67. (a) Find the values of $\alpha \in [0, 2\pi)$ that satisfy

$$\sin\alpha=-\frac{1}{2}\sqrt{3}$$

(b) Find the values of $\alpha \in [0, 2\pi)$ that satisfy

$$\tan\alpha=\sqrt{3}$$

68. (a) Find the values of $\alpha \in [0, 2\pi)$ that satisfy

$$\cos\alpha=-\frac{1}{2}\sqrt{2}$$

(b) Find the values of $\alpha \in [0, 2\pi)$ that satisfy

$$\sec\alpha=2$$

69. Show that the identity

$$1+\tan^2\theta=\sec^2\theta$$

follows from

$$\sin^2\theta+\cos^2\theta=1$$

70. Show that the identity

$$1+\cot^2\theta=\csc^2\theta$$

follows from

$$\sin^2\theta+\cos^2\theta=1$$

71. Solve $2\cos\theta\sin\theta=\sin\theta$ on $[0, 2\pi)$.

72. Solve $\sec^2 x=\sqrt{3}\tan x+1$ on $[0, \pi)$.

■ 1.1.5

73. Evaluate the following exponential expressions:
(a) $4^3 4^{-2/3}$ (b) $\frac{3^2 3^{1/2}}{3^{-1/2}}$ (c) $\frac{5^k 5^{2k-1}}{5^{1-k}}$

74. Evaluate the following exponential expressions:
(a) $(2^4 2^{-3/2})^2$ (b) $\left(\frac{6^{5/2}6^{2/3}}{6^{1/3}}\right)^3$ (c) $\left(\frac{3^{-2k+3}}{3^{4+k}}\right)^3$

75. Which real number x satisfies
(a) $\log_4 x=-2$? (b) $\log_{1/3} x=-3$? (c) $\log_{10} x=-2$?

76. Which real number x satisfies
(a) $\log_{1/2} x=-4$? (b) $\log_{1/4} x=2$? (c) $\log_5 x=3$?

77. Which real number x satisfies
(a) $\log_{1/2} 32=x$? (b) $\log_{1/3} 81=x$? (c) $\log_{10} 0.001=x$?

78. Which real number x satisfies
(a) $\log_4 64=x$? (b) $\log_{1/5} 625=x$? (c) $\log_{10} 10,000=x$?

79. Simplify the following expressions:
(a) $-\ln\frac{1}{3}$ (b) $\log_4(x^2-4)$ (c) $\log_2 4^{3x-1}$

80. Simplify the following expressions:
(a) $-\ln\frac{1}{5}$ (b) $\ln\frac{x^2-y^2}{\sqrt{x}}$ (c) $\log_3 3^{2x+1}$

81. Solve for x.
(a) $e^{3x-1}=2$ (b) $e^{-2x}=10$ (c) $e^{x^2-1}=10$

82. Solve for x.
(a) $3^x=81$ (b) $9^{2x+1}=27$ (c) $10^{5x}=1000$

83. Solve for x.
(a) $\ln(x-3)=5$ (b) $\ln(x+2)+\ln(x-2)=1$
(c) $\log_3 x^2-\log_3 2x=2$

84. Solve for x.
(a) $\ln(2x-3)=0$ (b) $\log_2(1-x)=3$
(c) $\ln x^3-2\ln x=1$

■ 1.1.6

In Problems 85–92, simplify each expression and write it in the standard form a + bi.

85. $(3-2i)-(-2+5i)$
86. $(7+i)-4$
87. $(4-2i)+(9+4i)$
88. $(6-4i)+(2+5i)$
89. $3(5+3i)$
90. $(2-3i)(5+2i)$
91. $(6-i)(6+i)$
92. $(-4-3i)(4+2i)$

In Problems 93–98, let $z=3-2i$, $u=-4+3i$, $v=3+5i$, and $w=1-i$. Compute the following expressions:

93. $\overline{z}$
94. $z+u$
95. $\overline{z+v}$
96. $\overline{v-w}$
97. $\overline{vw}$
98. $\overline{uz}$

99. If $z=a+bi$, find $z+\overline{z}$ and $z-\overline{z}$.

100. If $z=a+bi$, find $\overline{z}$. Use your answer to compute $\overline{(\overline{z})}$, and compare your answer with z.

In Problems 101–106, solve each quadratic equation in the complex number system.

101. $2x^2-3x+2=0$
102. $3x^2-2x+1=0$
103. $-x^2+x+2=0$
104. $-2x^2+x+3=0$
105. $4x^2-3x+1=0$
106. $-2x^2+4x-3=0$

In Problems 107–112, first determine whether the solutions of each quadratic equation are real or complex without solving the equation. Then solve the equation.

107. $3x^2-4x-7=0$
108. $3x^2-4x+7=0$
109. $-x^2+2x-1=0$
110. $4x^2-x+1=0$
111. $3x^2-5x+6=0$
112. $-x^2+7x-2=0$

113. Show $\overline{(\overline{z})}=z$.

114. Show $\overline{z+w}=\overline{z}+\overline{w}$.

115. Show $\overline{zw}=\overline{z}\,\overline{w}$.

Same #

EXAMPLE 16 Compare

$$f(x) = 3\sin\left(\frac{\pi}{4}x\right) \quad \text{and} \quad g(x) = \sin x$$

Solution The amplitude of $f(x)$ is 3, whereas the amplitude of $g(x)$ is 1. The period p of $f(x)$ satisfies $\frac{\pi}{4}p = 2\pi$ or $p = 8$, whereas the period of $g(x)$ is 2π. Graphs of $f(x)$ and $g(x)$ are shown in Figure 1.40. ■

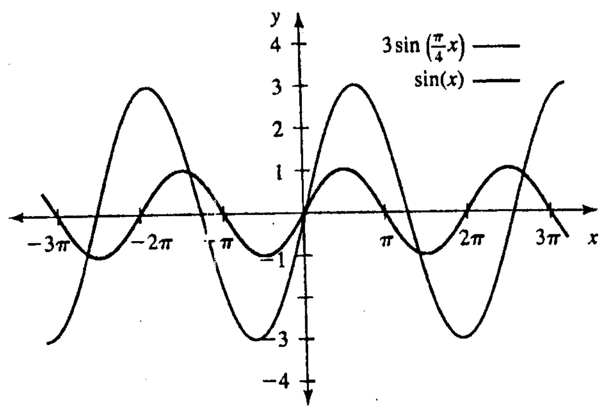

Figure 1.40 The graphs of $y = 3\sin(\frac{\pi}{4}x)$ and $g(x) = \sin x$ in Example 16.

Remark. A number is called **algebraic** if it is the solution of a polynomial equation with rational coefficients. For instance, $\sqrt{2}$ is algebraic, as it satisfies the equation $x^2 - 2 = 0$. Numbers that are not algebraic are called **transcendental.** For instance, π and e are transcendental.

A similar distinction is made for functions. We call a function $y = f(x)$ algebraic if it is the solution of an equation of the form

$$P_n(x)y^n + \cdots + P_1(x)y + P_0(x) = 0$$

in which the coefficients are polynomial functions in x with rational coefficients. For instance, the function $y = 1/(1+x)$ is algebraic, as it satisfies the equation $(x+1)y - 1 = 0$. Here, $P_1(x) = x + 1$ and $P_0(x) = -1$. Other examples of algebraic functions are polynomial functions with rational coefficients and rational functions with rational coefficients.

Functions that are not algebraic are called transcendental. All the trigonometric, exponential, and logarithmic functions that we introduced in this section are transcendental functions.

Section 1.2 Problems

■ 1.2.1

In Problems 1–4, state the range for the given functions. Graph each function.

1. $f(x) = x^2, x \in \mathbf{R}$ **2.** $f(x) = x^2, x \in [0, 1]$

3. $f(x) = x^2, -1 < x \leq 0$ **4.** $f(x) = x^2, -\frac{1}{2} < x < \frac{1}{2}$

5. (a) Show that, for $x \neq 1$,

$$\frac{x^2-1}{x-1} = x + 1$$

(b) Are the functions

$$f(x) = \frac{x^2-1}{x-1}, \quad x \neq 1$$

and

$$g(x) = x + 1, \quad x \in \mathbf{R}$$

equal?

6. (a) Show that

$$2|x-1| = \begin{cases} 2(x-1) & \text{for } x \geq 1 \\ 2(1-x) & \text{for } x \leq 1 \end{cases}$$

(b) Are the functions

$$f(x) = \begin{cases} 2-2x, & \text{for } 0 \le x \le 1 \\ 2x-2 & \text{for } 1 \le x \le 2 \end{cases}$$

and

$$g(x) = 2|x-1|, \; x \in [0,2]$$

equal?

In Problems 7–12, sketch the graph of each function and decide in each case whether the function is (i) even, (ii) odd, or (iii) does not show any obvious symmetry. Then use the criteria in Subsection 1.2.1 to check your answers.

7. $f(x) = 2x$
8. $f(x) = 3x^2$
9. $f(x) = |3x|$
10. $f(x) = 2x + 1$
11. $f(x) = -|x|$
12. $f(x) = 3x^3$

13. Suppose that

$$f(x) = x^2, \quad x \in \mathbf{R}$$

and

$$g(x) = 3 + x, \quad x \in \mathbf{R}$$

(a) Show that

$$(f \circ g)(x) = (3+x)^2, \quad x \in \mathbf{R}$$

(b) Show that

$$(g \circ f)(x) = 3 + x^2, \quad x \in \mathbf{R}$$

14. Suppose that

$$f(x) = x^3, \quad x \in \mathbf{R}$$

and

$$g(x) = 1 - x, \quad x \in \mathbf{R}$$

(a) Show that

$$(f \circ g)(x) = (1-x)^3, \quad x \in \mathbf{R}$$

(b) Show that

$$(g \circ f)(x) = 1 - x^3, \quad x \in \mathbf{R}$$

15. Suppose that

$$f(x) = 1 - x^2, \quad x \in \mathbf{R}$$

and

$$g(x) = 2x, \quad x \ge 0$$

(a) Find

$$(f \circ g)(x)$$

together with its domain.

(b) Find

$$(g \circ f)(x)$$

together with its domain.

16. Suppose that

$$f(x) = \frac{1}{x+1}, \quad x \ne -1$$

and

$$g(x) = 2x^2, \quad x \in \mathbf{R}$$

(a) Find $(f \circ g)(x)$. **(b)** Find $(g \circ f)(x)$.
In both (a) and (b), find the domain.

17. Suppose that

$$f(x) = 3x^2, \quad x \ge 3$$

and

$$g(x) = \sqrt{x}, \quad x \ge 0$$

Find $(f \circ g)(x)$ together with its domain.

18. Suppose that

$$f(x) = x^4, \quad x \ge 3$$

and

$$g(x) = \sqrt{x+1}, \quad x \ge 3$$

Find $(f \circ g)(x)$ together with its domain.

19. Suppose that $f(x) = x^2, x \ge 0$, and $g(x) = \sqrt{x}, x \ge 0$. Typically, $f \circ g \ne g \circ f$, but this is an example in which the order of composition does not matter. Show that $f \circ g = g \circ f$.

20. Suppose that $f(x) = x^4, x \ge 0$. Find $g(x)$ so that $f \circ g = g \circ f$.

■ 1.2.2

21. Use a graphing calculator to graph $f(x) = x^2, x \ge 0$, and $g(x) = x^4, x \ge 0$, together. For which values of x is $f(x) > g(x)$, and for which is $f(x) < g(x)$?

22. Use a graphing calculator to graph $f(x) = x^3, x \ge 0$, and $g(x) = x^5, x \ge 0$, together. When is $f(x) > g(x)$, and when is $f(x) < g(x)$?

23. Graph $y = x^n, x \ge 0$, for $n = 1, 2, 3$, and 4 in one coordinate system. Where do the curves intersect?

24. **(a)** Graph $f(x) = x, x \ge 0$, and $g(x) = x^2, x \ge 0$, together, in one coordinate system.

(b) For which values of x is $f(x) \ge g(x)$, and for which values of x is $f(x) \le g(x)$?

25. **(a)** Graph $f(x) = x^2$ and $g(x) = x^3$ for $x \ge 0$, together, in one coordinate system.

(b) Show algebraically that

$$x^2 \ge x^3$$

for $0 \le x \le 1$.

(c) Show algebraically that

$$x^2 \le x^3$$

for $x \ge 1$.

26. Show algebraically that if $n \ge m$,

$$x^n \le x^m \quad \text{for } 0 \le x \le 1$$

and

$$x^n \ge x^m \quad \text{for } x \ge 1$$

27. **(a)** Show that $y = x^2, x \in \mathbf{R}$, is an even function.

(b) Show that $y = x^3, x \in \mathbf{R}$, is an odd function.

28. Show that

(a) $y = x^n, x \in \mathbf{R}$, is an even function when n is an even integer.

(b) $y = x^n, x \in \mathbf{R}$, is an odd function when n is an odd integer.

29. In Example 5 of this section, we considered the chemical reaction

$$A + B \to AB$$

Assume that initially only A and B are in the reaction vessel and that the initial concentrations are $a = [A] = 3$ and $b = [B] = 4$.

(a) We found that the reaction rate $R(x)$, where x is the concentration of AB, is given by

$$R(x) = k(a - x)(b - x)$$

where a is the initial concentration of A, b is the initial concentration of B, and k is the constant of proportionality. Suppose that the reaction rate $R(x)$ is equal to 9 when the concentration of AB is $x = 1$. Use this relationship to find the reaction rate $R(x)$.

(b) Determine the appropriate domain of $R(x)$, and use a graphing calculator to sketch the graph of $R(x)$.

30. An autocatalytic reaction uses its resulting product for the formation of a new product, as in the reaction

$$A + X \rightarrow X$$

If we assume that this reaction occurs in a closed vessel, then the reaction rate is given by

$$R(x) = kx(a - x)$$

for $0 \le x \le a$, where a is the initial concentration of A and x is the concentration of X.

(a) Show that $R(x)$ is a polynomial and determine its degree.

(b) Graph $R(x)$ for $k = 2$ and $a = 6$. Find the value of x at which the reaction rate is maximal.

31. Suppose that a beetle walks up a tree along a straight line at a constant speed of 1 meter per hour. What distance will the beetle have covered after 1 hour, 2 hours, and 3 hours? Write an equation that expresses the distance (in meters) as a function of the time (in hours), and show that this function is a polynomial of degree 1.

32. Suppose that a fungal disease originates in the middle of an orchard, initially affecting only one tree. The disease spreads out radially at a constant speed of 10 feet per day. What area will be affected after 2 days, 4 days, and 8 days? Write an equation that expresses the affected area as a function of time, measured in days, and show that this function is a polynomial of degree 2.

■ 1.2.3

In Problems 33–36, for each function, find the largest possible domain and determine the range.

33. $f(x) = \dfrac{1}{1-x}$ **34.** $f(x) = \dfrac{2x}{(x-2)(x+3)}$

35. $f(x) = \dfrac{x-2}{x^2-9}$ **36.** $f(x) = \dfrac{1}{x^2+1}$

37. Compare $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ for $x > 0$ by graphing the two functions. Where do the curves intersect? Which function is greater for small values of x? for large values of x?

38. Let n and m be two positive integers with $m \le n$. Answer the following questions about $y = x^{-n}$ and $y = x^{-m}$ for $x > 0$: Where do the curves intersect? Which function is greater for small values of x? for large values of x?

39. Let

$$f(x) = \frac{1}{x+1}, \quad x > -1$$

(a) Use a graphing calculator to graph $f(x)$.

(b) On the basis of the graph in (a), determine the range of $f(x)$.

(c) For which values of x is $f(x) = 2$?

(d) On the basis of the graph in (a), determine how many solutions $f(x) = a$ has, where a is in the range of $f(x)$.

40. Let

$$f(x) = \frac{2x}{3+x}, \quad x \ge 0$$

(a) Use a graphing calculator to graph $f(x)$.

(b) Find the range of $f(x)$.

(c) For which values of x is $f(x) = 1$?

(d) Based on the graph in (a), explain in words why, for any value a in the range of $f(x)$, you can find exactly one value $x \ge 0$ such that $f(x) = a$. Determine x by solving $f(x) = a$.

41. Let

$$f(x) = \frac{3x}{1+x}, \quad x \ge 0$$

(a) Use a graphing calculator to graph $f(x)$.

(b) Find the range of $f(x)$.

(c) For which values of x is $f(x) = 2$?

(d) On the basis of the graph in (a), explain in words why, for any value a in the range of $f(x)$, you can find exactly one value $x \ge 0$ such that $f(x) = a$. Determine x by solving $f(x) = a$.

In Problems 42–44, we discuss the Monod growth function, which was introduced in Example 6 of this section.

42. Use a graphing calculator to investigate the Monod growth function

$$r(N) = \frac{aN}{k+N}, \quad N \ge 0$$

where a and k are positive constants.

(a) Graph $r(N)$ for (i) $a = 5$ and $k = 1$, (ii) $a = 5$ and $k = 3$, and (iii) $a = 8$ and $k = 1$. Place all three graphs in one coordinate system.

(b) On the basis of the graphs in (a), describe in words what happens when you change a.

(c) On the basis of the graphs in (a), describe in words what happens when you change k.

43. The Monod growth function $r(N)$ describes growth as a function of nutrient concentration N. Assume that

$$r(N) = 5\frac{N}{1+N}, \quad N \ge 0$$

Find the percentage increase when the nutrient concentration is doubled from $N = 0.1$ to $N = 0.2$. Compare this result with what you find when you double the nutrient concentration from $N = 10$ to $N = 20$. This is an example of *diminishing return*.

44. The Monod growth function $r(N)$ describes growth as a function of nutrient concentration N. Assume that

$$r(N) = a\frac{N}{k+N}, \quad N \ge 0$$

where a and k are positive constants.

(a) What happens to $r(N)$ as N increases? Use this relationship to explain why a is called the saturation level.

(b) Show that k is the half-saturation constant; that is, show that if $N = k$, then $r(N) = a/2$.

45. Let

$$f(x) = \frac{x^2}{4+x^2}, \quad x \ge 0$$

(a) Use a graphing calculator to graph $f(x)$.

(b) On the basis of your graph in (a), find the range of $f(x)$.

(c) What happens to $f(x)$ as x gets larger?

46. The function

$$f(x) = \frac{x^n}{b^n + x^n}, \quad x \geq 0$$

where n is a positive integer and b is a positive real number, is used in biochemistry to model reaction rates as a function of the concentration of some reactants.

(a) Use a graphing calculator to graph $f(x)$ for $n = 1, 2$, and 3 in one coordinate system when $b = 2$.

(b) Where do the three graphs in (a) intersect?

(c) What happens to $f(x)$ as x gets larger?

(d) For an arbitrary positive value of b, show that $f(b) = 1/2$. On the basis of this demonstration and your answer in (c), explain why b is called the half-saturation constant.

■ 1.2.4

In Problems 47–50, use a graphing calculator to sketch the graphs of the functions.

47. $y = x^{3/2}, x \geq 0$

48. $y = x^{1/3}, x \geq 0$

49. $y = x^{-1/4}, x > 0$

50. $y = 2x^{-7/8}, x > 0$

51. (a) Graph $y = x^{-1/2}, x > 0$, and $y = x^{1/2}, x \geq 0$, together, in one coordinate system.

(b) Show algebraically that

$$x^{-1/2} \geq x^{1/2}$$

for $0 < x \leq 1$.

(c) Show algebraically that

$$x^{-1/2} \leq x^{1/2}$$

for $x \geq 1$.

52. (a) Graph $y = x^{5/2}, x \geq 0$, and $y = x^{1/2}, x \geq 0$, together, in one coordinate system.

(b) Show algebraically that

$$x^{5/2} \leq x^{1/2}$$

for $0 \leq x \leq 1$. (*Hint*: Show that $x^{1/2}/x^{-1/2} = x \leq 1$ for $0 < x \leq 1$.)

(c) Show algebraically that

$$x^{5/2} \geq x^{1/2}$$

for $x \geq 1$.

In Problems 53–56, sketch each scaling relation (Niklas, 1994).

53. In a sample based on 46 species, leaf area was found to be proportional to (stem diameter)$^{1.84}$. On the basis of your graph, as stem diameter increases, does leaf area increase or decrease?

54. In a sample based on 28 species, the volume fraction of spongy mesophyll was found to be proportional to (leaf thickness)$^{-0.49}$ (The spongy mesophyll is part of the internal tissue of a leaf blade.) On the basis of your graph, as leaf thickness increases, does the volume fraction of spongy mesophyll increase or decrease?

55. In a sample of 60 species of trees, wood density was found to be proportional to (breaking strength)$^{0.82}$. On the basis of your graph, does breaking strength increase as wood density increases? or as wood density decreases?

56. Suppose that a cube of length L and volume V has mass M and that $M = 0.35V$. How does the length of the cube depend on its mass?

■ 1.2.5

57. Assume that a population size at time t is $N(t)$ and that

$$N(t) = 2^t, \quad t \geq 0$$

(a) Find the population size for $t = 0, 1, 2, 3$, and 4.

(b) Graph $N(t)$ for $t \geq 0$.

58. Assume that a population size at time t is $N(t)$ and that

$$N(t) = 40 \cdot 2^t, \quad t \geq 0$$

(a) Find the population size at time $t = 0$.

(b) Show that

$$N(t) = 40e^{t \ln 2}, \quad t \geq 0$$

(c) How long will it take until the population size reaches 1000? [*Hint*: Find t so that $N(t) = 1000$.]

59. The half-life of C^{14} is 5730 years. If a sample of C^{14} has a mass of 20 micrograms at time $t = 0$, how much is left after 2000 years?

60. The half-life of C^{14} is 5730 years. If a sample of C^{14} has a mass of 20 micrograms at time 0, how long will it take until (a) 10 grams and (b) 5 grams are left?

61. After 7 days, a particular radioactive substance decays to half of its original amount. Find the decay rate of this substance.

62. After 5 days, a particular radioactive substance decays to 37% of its original amount. Find the half-life of this substance.

63. Polonium 210 (Po^{210}) has a half-life of 140 days.

(a) If a sample of Po^{210} has a mass of 300 micrograms, find a formula for the mass after t days.

(b) How long would it take this sample to decay to 20% of its original amount?

(c) Sketch the graph of the amount of mass left after t days.

64. The half-life of C^{14} is 5730 years. Suppose that wood found at an archeological excavation site contains about 35% as much C^{14} (in relation to C^{12}) as does living plant material. Determine when the wood was cut.

65. The half-life of C^{14} is 5730 years. Suppose that wood found at an archeological excavation site is 15,000 years old. How much C^{14} (based on C^{12} content) does the wood contain relative to living plant material?

66. The age of rocks of volcanic origin can be estimated with isotopes of argon 40 (Ar^{40}) and potassium 40 (K^{40}). K^{40} decays into Ar^{40} over time. If a mineral that contains potassium is buried under the right circumstances, argon forms and is trapped. Since argon is driven off when the mineral is heated to very high temperatures, rocks of volcanic origin do not contain argon when they are formed. The amount of argon found in such rocks can therefore be used to determine the age of the rock. Assume that a sample of volcanic rock contains 0.00047% K^{40}. The sample also contains 0.000079% Ar^{40}. How old is the rock? (The decay rate of K^{40} to Ar^{40} is 5.335×10^{-10}/yr.)

67. *(Adapted from Moss, 1980)* Hall (1964) investigated the change in population size of the zooplankton species *Daphnia galeata mendota* in Base Line Lake, Michigan. The population size $N(t)$ at time t was modeled by the equation

$$N(t) = N_0 e^{rt}$$

where N_0 denotes the population size at time 0. The constant r is called the **intrinsic rate of growth.**

(a) Plot $N(t)$ as a function of t if $N_0 = 100$ and $r = 2$. Compare your graph against the graph of $N(t)$ when $N_0 = 100$ and $r = 3$. Which population grows faster?

(b) The constant r is an important quantity because it describes how quickly the population changes. Suppose that you determine the size of the population at the beginning and at the end of a period of length 1, and you find that at the beginning there were 200 individuals and after one unit of time there were 250 individuals. Determine r. [*Hint*: Consider the ratio $N(t+1)/N(t)$.]

68. Fish are indeterminate growers; that is, they grow throughout their lifetime. The growth of fish can be described by the von Bertalanffy function

$$L(x) = L_\infty(1 - e^{-kx})$$

for $x \geq 0$, where $L(x)$ is the length of the fish at age x and k and L_∞ are positive constants.

(a) Use a graphing calculator to graph $L(x)$ for $L_\infty = 20$, for (i) $k = 1$ and (ii) $k = 0.1$.

(b) For $k = 1$, find x so that the length is 90% of L_∞. Repeat for 99% of L_∞. Can the fish ever attain length L_∞? Interpret the meaning of L_∞.

(c) Compare the graphs obtained in (a). Which growth curve reaches 90% of L_∞ faster? Can you explain what happens to the curve of $L(x)$ when you vary k (for fixed L_∞)?

■ 1.2.6

69. Which of the following functions is one to one (use the horizontal line test)?

(a) $f(x) = x^2, x \geq 0$ (b) $f(x) = x^2, x \in \mathbf{R}$

(c) $f(x) = \frac{1}{x}, x > 0$ (d) $f(x) = e^x, x \in \mathbf{R}$

(e) $f(x) = \frac{1}{x^2}, x \neq 0$ (f) $f(x) = \frac{1}{x^2}, x > 0$

70. (a) Show that $f(x) = x^3 - 1, x \in \mathbf{R}$, is one to one, and find its inverse together with its domain.

(b) Graph $f(x)$ and $f^{-1}(x)$ in one coordinate system, together with the line $y = x$, and convince yourself that the graph of $f^{-1}(x)$ can be obtained by reflecting the graph of $f(x)$ about the line $y = x$.

71. (a) Show that $f(x) = x^2 + 1, x \geq 0$, is one to one, and find its inverse together with its domain.

(b) Graph $f(x)$ and $f^{-1}(x)$ in one coordinate system, together with the line $y = x$, and convince yourself that the graph of $f^{-1}(x)$ can be obtained by reflecting the graph of $f(x)$ about the line $y = x$.

72. (a) Show that $f(x) = \sqrt{x}, x \geq 0$, is one to one, and find its inverse together with its domain.

(b) Graph $f(x)$ and $f^{-1}(x)$ in one coordinate system, together with the line $y = x$, and convince yourself that the graph of $f^{-1}(x)$ can be obtained by reflecting the graph of $f(x)$ about the line $y = x$.

73. (a) Show that $f(x) = 1/x^3, x > 0$, is one to one, and find its inverse together with its domain.

(b) Graph $f(x)$ and $f^{-1}(x)$ in one coordinate system, together with the line $y = x$, and convince yourself that the graph of $f^{-1}(x)$ can be obtained by reflecting the graph of $f(x)$ about the line $y = x$.

74. The reciprocal of a function $f(x)$ can be written as either $1/f(x)$ or $[f(x)]^{-1}$. The point of this problem is to make clear that a reciprocal of a function has nothing to do with the inverse of a function. As an example, let $f(x) = 2x + 1, x \in \mathbf{R}$. Find both $[f(x)]^{-1}$ and $f^{-1}(x)$, and compare the two functions. Graph all three functions together.

■ 1.2.7

75. Find the inverse of $f(x) = 3^x, x \in \mathbf{R}$, together with its domain, and graph both functions in the same coordinate system.

76. Find the inverse of $f(x) = 5^x, x \in \mathbf{R}$, together with its domain, and graph both functions in the same coordinate system.

77. Find the inverse of $f(x) = (\frac{1}{4})^x, x \in \mathbf{R}$, together with its domain, and graph both functions in the same coordinate system.

78. Find the inverse of $f(x) = (\frac{1}{3})^x, x \in \mathbf{R}$, together with its domain, and graph both functions in the same coordinate system.

79. Find the inverse of $f(x) = 2^x, x \geq 0$, together with its domain, and graph both functions in the same coordinate system.

80. Find the inverse of $f(x) = (\frac{1}{2})^x, x \geq 0$, together with its domain, and graph both functions in the same coordinate system.

81. Simplify the following expressions:

(a) $2^{5\log_2 x}$ (b) $3^{4\log_3 x}$

(c) $5^{5\log_{1/5} x}$ (d) $4^{-2\log_2 x}$

(e) $2^{3\log_{1/2} x}$ (f) $4^{-\log_{1/2} x}$

82. Simplify the following expressions:

(a) $\log_4 16^x$ (b) $\log_2 16^x$

(c) $\log_3 27^x$ (d) $\log_{1/2} 4^x$

(e) $\log_{1/2} 8^{-x}$ (f) $\log_3 9^{-x}$

83. Simplify the following expressions:

(a) $\ln x^2 + \ln x^3$ (b) $\ln x^4 - \ln x^{-2}$

(c) $\ln(x^2 - 1) - \ln(x + 1)$ (d) $\ln x^{-1} + \ln x^{-3}$

84. Simplify the following expressions:

(a) $e^{3\ln x}$ (b) $e^{-\ln(x^2+1)}$

(c) $e^{-2\ln(1/x)}$ (d) $e^{-2\ln x}$

85. Write the following expressions in terms of base e, and simplify:

(a) 3^x (b) 4^{x^2-1} (c) 2^{-x-1} (d) 3^{-4x+1}

86. Write the following expressions in terms of base e:

(a) $\log_2(x^2 - 1)$ (b) $\log_3(5x + 1)$

(c) $\log(x + 2)$ (d) $\log_2(2x^2 - 1)$

87. Show that the function $y = (1/2)^x$ can be written in the form $y = e^{-\mu x}$, where μ is a positive constant. Determine μ.

88. Show that if $0 < a < 1$, then the function $y = a^x$ can be written in the form $y = e^{-\mu x}$, where μ is a positive constant. Write μ in terms of a.

89. Assume that two DNA sequences of common origin, each of length 300 nucleotides, differ from each other by 47 nucleotides. Use the Jukes and Cantor correction of Example 15 to find an estimate for the number K of substitutions per site.

90. A community measure that takes both species abundance and species richness into account is the Shannon diversity index H. To calculate H, the proportion p_i of species i in the community is used. Assume that the community consists of S species. Then

$$H = -(p_1 \ln p_1 + p_2 \ln p_2 + \cdots + p_S \ln p_S)$$

(a) Assume that $S = 5$ and that all species are equally abundant; that is, $p_1 = p_2 = \cdots = p_5$. Compute H.

(b) Assume that $S = 10$ and that all species are equally abundant; that is, $p_1 = p_2 = \cdots = p_{10}$. Compute H.

(c) A measure of equitability (or evenness) of the species distribution can be measured by dividing the diversity index H by $\ln S$. Compute $H/\ln S$ for $S = 5$ and $S = 10$.

(d) Show that, in general, if there are N species and all species are equally abundant, then

$$\frac{H}{\ln S} = 1$$

■ 1.2.8

In Problems 91–96, for each given pair of functions, use a graphing calculator to compare the functions. Describe what you see.

91. $y = \sin x$ and $y = 2\sin x$
92. $y = \sin x$ and $y = \sin(2x)$
93. $y = \cos x$ and $y = 2\cos x$
94. $y = \cos x$ and $y = \cos(2x)$
95. $y = \tan x$ and $y = 2\tan x$
96. $y = \tan x$ and $y = \tan(2x)$
97. Let
$$f(x) = 3\sin(4x), \quad x \in \mathbf{R}$$
Find the amplitude and the period of $f(x)$.
98. Let
$$f(x) = -2\sin\left(\frac{x}{2}\right), \quad x \in \mathbf{R}$$
Find the amplitude and the period of $f(x)$.
99. Let
$$f(x) = 4\sin(2\pi x), \quad x \in \mathbf{R}$$
Find the amplitude and the period of $f(x)$.
100. Let
$$f(x) = -\frac{3}{2}\sin\left(\frac{\pi}{3}x\right), \quad x \in \mathbf{R}$$
Find the amplitude and the period of $f(x)$.
101. Let
$$f(x) = 4\cos\left(\frac{x}{4}\right), \quad x \in \mathbf{R}$$
Find the amplitude and the period of $f(x)$.
102. Let
$$f(x) = 7\cos(2x), \quad x \in \mathbf{R}$$
Find the amplitude and the period of $f(x)$.
103. Let
$$f(x) = -3\cos\left(\frac{\pi x}{5}\right), \quad x \in \mathbf{R}$$
Find the amplitude and the period of $f(x)$.
104. Let
$$f(x) = -\frac{2}{3}\cos\left(\frac{3x}{\pi}\right), \quad x \in \mathbf{R}$$
Find the amplitude and the period of $f(x)$.
105. Use the fact that $\sec x = \frac{1}{\cos x}$ to explain why the maximum domain of $y = \sec x$ consists of all real numbers except odd integer multiples of $\pi/2$.
106. Use the fact that $\csc x = \frac{1}{\sin x}$ to explain why the maximum domain of $y = \csc x$ consists of all real numbers except integer multiples of π.

new

■ 1.3 Graphing

In the preceding section, we introduced the functions most important to our study. You must be able to graph the following functions without a calculator: $y = c, x, x^2, x^3, 1/x, e^x, \ln x, \sin x, \cos x, \sec x$, and $\tan x$. This will help you to sketch functions quickly and to come up with an analytical description of a function based on a graph. In this section, you will learn how to obtain new functions from these basic functions and how to graph them. In addition, we will introduce important transformations that are often used to display data graphically.

■ 1.3.1 Graphing and Basic Transformations of Functions

In this subsection, we will recall some basic transformations: vertical and horizontal translations, reflections about $x = 0$ and $y = 0$, and stretching and compressing.

Definition The graph of
$$y = f(x) + a$$
is a **vertical translation** of the graph of $y = f(x)$. If $a > 0$, the graph of $y = f(x)$ is shifted up a units; if $a < 0$, the graph of $y = f(x)$ is shifted down $|a|$ units.

This definition is illustrated in Figure 1.41, where we display $y = x^2$, $y = x^2 + 2$, and $y = x^2 - 2$.

Definition The graph of
$$y = f(x - c)$$
is a **horizontal translation** of the graph of $y = f(x)$. If $c > 0$, the graph of $y = f(x)$ is shifted c units to the right; if $c < 0$, the graph of $y = f(x)$ is shifted $|c|$ units to the left.

The next example presents a response that depends on two independent variables, and shows how to draw a graph of this more complex relationship.

EXAMPLE 13

The successful germination of seeds depends on both temperature and humidity. When the humidity is too low, seeds tend not to germinate at all, regardless of the temperature. Germination success is highest for intermediate values of temperature. Finally, seeds tend to germinate better when humidity levels are higher.

One way to translate this information into a graph is to graph germination success as a function of temperature for different levels of humidity. If we measure temperature in Fahrenheit or Celsius, we can restrict the graphs to the first quadrant (Figure 1.69), since the temperature needs to be well above freezing for germination to occur (the temperature at which freezing starts is 0°C, or 32°F). Germination success will be between 0 and 100%. To sketch the graphs, it is better not to label the axes beyond what we provided in Figure 1.69, because we do not know the exact numerical response.

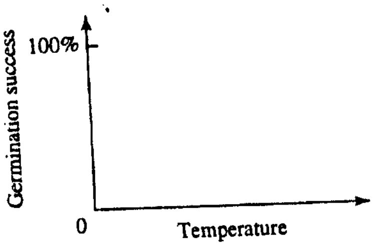

Figure 1.69 The coordinate system for germination success as a function of temperature can be restricted to the first quadrant. Germination success will be between 0 and 100%.

There is enough information to provide three graphs: one for low humidity, one for intermediate humidity, and one for high humidity. We will graph them all in one coordinate system, so that it is easier to compare the different responses. The graph for low humidity is a horizontal line where germination success is 0%. For intermediate and high humidity, the graphs are hump shaped, since germination success is highest for intermediate values of temperature. The graph for high humidity is above the graph for intermediate humidity, because seeds tend to germinate better when humidity levels are higher (Figure 1.70). ■

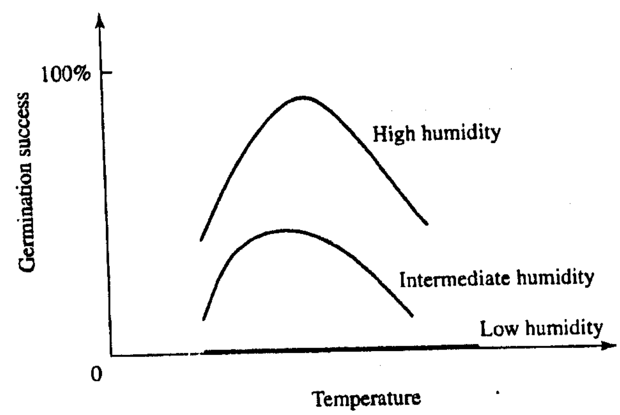

Figure 1.70 Germination success as a function of temperature for three humidity levels (low, intermediate, high).

Section 1.3 Problems

■ 1.3.1

In Problems 1–22, sketch the graph of each function. Do not use a graphing calculator. (Assume the largest possible domain.)

1. $y = x^2 + 1$

2. $y = -(x-2)^2 + 1$

3. $y = x^3 - 2$

4. $y = -x^4 + 1$

5. $y = -2x^2 - 3$

6. $y = -(3-x)^2$

7. $y = 3 + 1/x$

8. $y = 1 - 1/x$

9. $y = 1/(x-1)$

10. $y = 1 + 1/(x+2)^2$

11. $y = \exp(x-2)$

12. $y = \exp(-x)$

13. $y = e^{-(x+3)}$

14. $y = 3e^{2x+1}$

15. $y = \ln(x+1)$

16. $y = \ln(x-3)$

17. $y = -\ln(x-1) + 1$

18. $y = -\ln(1-x)$

19. $y = 2\sin(x + \pi/4)$

20. $y = 0.2\cos(-x)$

21. $y = -\sin(\pi x/2)$

22. $y = -2\cos(\pi x/4)$

23. Explain how the following functions can be obtained from $y = x^2$ by basic transformations:
(a) $y = x^2 - 2$ **(b)** $y = (x-1)^2 + 1$ **(c)** $y = -2(x+2)^2$

24. Explain how the following functions can be obtained from $y = x^3$ by basic transformations:
(a) $y = x^3 + 1$ **(b)** $y = (x+1)^3 - 1$ **(c)** $y = -3(x-2)^3$

25. Explain how the following functions can be obtained from $y = 1/x$ by basic transformations:
(a) $y = 1 - \frac{1}{x}$ **(b)** $y = -\frac{1}{x-1}$ **(c)** $y = \frac{x}{x+1}$

26. Explain how the following functions can be obtained from $y = 1/x^2$ by basic transformations:
(a) $y = \frac{1}{x^2} + 1$ (b) $y = -\frac{1}{(x+1)^2}$ (c) $y = -\frac{1}{x^2} - 2$

27. Explain how the following functions can be obtained from $y = e^x$ by basic transformations:
(a) $y = 2e^x - 1$ (b) $y = -e^{-x}$ (c) $y = e^{x-2} + 1$

28. Explain how the following functions can be obtained from $y = e^x$ by basic transformations:
(a) $y = e^{-x} - 1$ (b) $y = -e^x + 1$ (c) $y = -e^{x-3} - 2$

29. Explain how the following functions can be obtained from $y = \ln x$ by basic transformations:
(a) $y = \ln(x-1)$ (b) $y = -\ln x + 1$ (c) $y = \ln(x+3) - 1$

30. Explain how the following functions can be obtained from $y = \ln x$ by basic transformations:
(a) $y = \ln(1-x)$ (b) $y = \ln(2+x) - 1$
(c) $y = -\ln(2-x) + 1$

31. Explain how the following functions can be obtained from $y = \sin x$ by basic transformations:
(a) $y = 1 - \sin x$ (b) $y = \sin\left(x - \frac{\pi}{4}\right)$
(c) $y = -\sin\left(x + \frac{\pi}{3}\right)$

32. Explain how the following functions can be obtained from $y = \cos x$ by basic transformations:
(a) $y = 1 + 2\cos x$ (b) $y = -\cos\left(x + \frac{\pi}{4}\right)$
(c) $y = -\cos\left(\frac{\pi}{2} - x\right)$

■ 1.3.2

33. Find the following numbers on a number line that is on a logarithmic scale (base 10): 0.0002, 0.02, 1, 5, 50, 100, 1000, 8000, and 20000.

34. Find the following numbers on a number line that is on a logarithmic scale (base 10): 0.03, 0.7, 1, 2, 5, 10, 17, 100, 150, and 2000.

35. (a) Find the following numbers on a number line that is on a logarithmic scale (base 10): $10^2, 10^{-3}, 10^{-4}, 10^{-7}$, and 10^{-10}.
(b) Can you find 0 on a number line that is on a logarithmic scale?
(c) Can you find negative numbers on a number line that is on a logarithmic scale?

36. (a) Find the following numbers on a number line that is on a logarithmic scale (base 10):
(i) $10^{-3}, 2\times10^{-3}, 5\times10^{-3}$ (ii) $10^{-1}, 2\times10^{-1}, 5\times10^{-1}$
(iii) $10^2, 2\times10^2, 5\times10^2$
(b) From your answers to (a), how many units (on a logarithmic scale) is (i) 2×10^{-3} from 10^{-3} (ii) 2×10^{-1} from 10^{-1} and (iii) 2×10^2 from 10^2?
(c) From your answers to (a), how many units (on a logarithmic scale) is (i) 5×10^{-3} from 10^{-3} (ii) 5×10^{-1} from 10^{-1} and (iii) 5×10^2 from 10^2?

In Problems 37–42, insert an appropriate number in the blank space.

37. The longest known species of worms is the earthworm *Microchaetus rappi* of South Africa; in 1937, a 6.7-m-long specimen was collected from the Transvaal. The shortest worm is *Chaetogaster annandalei*, which measures less than 0.51 mm in length. *M. rappi* is ____ order(s) of magnitude longer than *C. annandalei*.

38. Both the La Plata river dolphin (*Pontoporia blainvillei*) and the sperm whale (*Physeter macrocephalus*) belong to the suborder Odontoceti (individuals that have teeth). A La Plata river dolphin weighs between 30 and 50 kg, whereas a sperm whale weighs between 35,000 and 40,000 kg. A sperm whale is ____ order(s) of magnitude heavier than a La Plata river dolphin.

39. Compare a ball of radius 1 cm against a ball of radius 10 cm. The radius of the larger ball is ____ order(s) of magnitude bigger than the radius of the smaller ball. The volume of the larger ball is ____ order(s) of magnitude bigger than the volume of the smaller ball.

40. Compare a square with side length 1 m against a square with side length 100 m. The area of the larger square is ____ order(s) of magnitude larger than the area of the smaller square.

41. The diameter of a typical bacterium is about 0.5 to 1 μm. An exception is the bacterium *Epulopiscium fishelsoni*, which is about 600 μm long and 80 μm wide. The volume of *E. fishelsoni* is about ____ order(s) of magnitude larger than that of a typical bacterium. (*Hint*: Approximate the shape of a typical bacterium by a sphere and the shape of *E. fishelsoni* by a cylinder.)

42. The length of a typical bacterial cell is about one-tenth that of a small eukaryotic cell. Consequently, the cell volume of a bacterium is about ____ order(s) of magnitude smaller than that of a small eukaryotic cell. (*Hint*: Approximate the shapes of both types of cells by spheres.)

■ 1.3.3

In Problems 43–46, when $\log y$ is graphed as a function of x, a straight line results. Graph straight lines, each given by two points, on a log-linear plot, and determine the functional relationship. (The original x–y coordinates are given.)

43. $(x_1, y_1) = (0, 5), (x_2, y_2) = (3, 1)$
44. $(x_1, y_1) = (-1, 4), (x_2, y_2) = (2, 8)$
45. $(x_1, y_1) = (-2, 3), (x_2, y_2) = (1, 1)$
46. $(x_1, y_1) = (1, 4), (x_2, y_2) = (6, 1)$

In Problems 47–54, use a logarithmic transformation to find a linear relationship between the given quantities and graph the resulting linear relationship on a log-linear plot.

47. $y = 3 \times 10^{-2x}$ 48. $y = 4 \times 10^{5x}$
49. $y = 2e^{-1.2x}$ 50. $y = 7e^{3x}$
51. $y = 5 \times 2^{4x}$ 52. $y = 6 \times 2^{-0.9x}$
53. $y = 4 \times 3^{2x}$ 54. $y = 5^{-6x}$

In Problems 55–58, when $\log y$ is graphed as a function of $\log x$, a straight line results. Graph straight lines, each given by two points, on a log-log plot, and determine the functional relationship. (The original x–y coordinates are given.)

55. $(x_1, y_1) = (1, 2), (x_2, y_2) = (5, 1)$
56. $(x_1, y_1) = (3, 5), (x_2, y_2) = (1, 5)$
57. $(x_1, y_1) = (4, 2), (x_2, y_2) = (8, 8)$
58. $(x_1, y_1) = (2, 5), (x_2, y_2) = (5, 2)$

In Problems 59–66, use a logarithmic transformation to find a linear relationship between the given quantities and graph the resulting linear relationship on a log-log plot.

59. $y = 2x^5$ 60. $y = 3x^2$
61. $y = x^6$ 62. $y = 5x^3$
63. $y = x^{-2}$ 64. $y = 6x^{-1}$
65. $y = 4x^{-3}$ 66. $y = 7x^{-5}$

In Problems 67–72, use a logarithmic transformation to find a linear relationship between the given quantities and determine whether a log-log or log-linear plot should be used to graph the resulting linear relationship.

67. $f(x) = 3x^{1.7}$ **68.** $g(s) = 1.8e^{-0.2s}$

69. $N(t) = 130 \times 2^{1.2t}$ **70.** $I(u) = 4.8u^{-0.89}$

71. $R(t) = 3.6t^{1.2}$ **72.** $L(c) = 1.7 \times 10^{2.3c}$

73. The following table is based on a functional relationship between x and y that is either an exponential or a power function:

x	y
1	1.8
2	2.07
4	2.38
10	2.85
20	3.28

Use an appropriate logarithmic transformation and a graph to decide whether the table comes from a power function or an exponential function, and find the functional relationship between x and y.

74. The following table is based on a functional relationship between x and y that is either an exponential or a power function:

x	y
0.5	7.81
1	3.4
1.5	2.09
2	1.48
2.5	1.13

Use an appropriate logarithmic transformation and a graph to decide whether the table comes from a power function or an exponential function, and find the functional relationship between x and y.

75. The following table is based on a functional relationship between x and y that is either an exponential or a power function:

x	y
−1	0.398
−0.5	1.26
0	4
0.5	12.68
1	40.18

Use an appropriate logarithmic transformation and a graph to decide whether the table comes from a power function or an exponential function, and find the functional relationship between x and y.

76. The following table is based on a functional relationship between x and y that is either an exponential or a power function:

x	y
0	3
0.5	2.20
1	1.61
1.5	1.18
2	0.862

Use an appropriate logarithmic transformation and a graph to decide whether the table comes from a power function or an exponential function, and find the functional relationship between x and y.

77. The following table is based on a functional relationship between x and y that is either an exponential or a power function:

x	y
0.1	0.045
0.5	1.33
1	5.7
1.5	13.36
2	24.44

Use an appropriate logarithmic transformation and a graph to decide whether the table comes from a power function or an exponential function, and find the functional relationship between x and y.

78. The following table is based on a functional relationship between x and y that is either an exponential or a power function:

x	y
0.1	1.72
0.5	1.41
1	1.11
1.5	0.872
2	0.685

Use an appropriate logarithmic transformation and a graph to decide whether the table comes from a power function or an exponential function, and find the functional relationship between x and y.

So far, we have always used base 10 for a logarithmic transformation. The reason for this is that our number system is based on base 10 and it is therefore easy to logarithmically transform numbers of the form ..., 0.01, 0.1, 1, 10, 100, 1000, ... when we use base 10. In Problems 79–82, use the indicated base to logarithmically transform each exponential relationship so that a linear relationship results. Then use the indicated base to graph each relationship in a coordinate system whose axes are accordingly transformed so that a straight line results.

79. $y = 2^x$; base 2 **80.** $y = 3^x$; base 3

81. $y = 2^{-x}$; base 2 **82.** $y = 3^{-x}$; base 3

83. Suppose that $N(t)$ denotes a population size at time t and satisfies the equation

$$N(t) = 2e^{3t} \quad \text{for } t \geq 0$$

(a) If you graph $N(t)$ as a function of t on a semilog plot, a straight line results. Explain why.

(b) Graph $N(t)$ as a function of t on a semilog plot, and determine the slope of the resulting straight line.

84. Suppose that you follow a population over time. When you plot your data on a semilog plot, a straight line with slope 0.03 results. Furthermore, assume that the population size at time 0 was 20. If $N(t)$ denotes the population size at time t, what function best describes the population size at time t?

85. Species–Area Curves Many studies have shown that the number of species on an island increases with the area of the island. Frequently, the functional relationship between the number of species (S) and the area (A) is approximated by $S = CA^z$, where z is a constant that depends on the particular species and habitat in the study. (Actual values of z range from about 0.2 to 0.35.) Suppose that the best fit to your data points on a log-log scale is a straight line. Is your model $S = CA^z$ an appropriate description of your data? If yes, how would you find z?

86. Michaelis–Menten Equation Enzymes serve as catalysts in many chemical reactions in living systems. The simplest such reactions transform a single substrate into a product with the help of an enzyme. The Michaelis–Menten equation describes the initial velocity of such enzymatically controlled reactions. The equation, which gives the relationship between the initial velocity of the reaction (v_0) and the concentration of the substrate (s_0), is

$$v_0 = \frac{v_{max} s_0}{s_0 + K_m}$$

where v_{max} is the maximum velocity at which the product may be formed and K_m is the Michaelis–Menten constant. Note that this equation has the same form as the Monod growth function.

(a) Show that the Michaelis–Menten equation can be written in the form

$$\frac{1}{v_0} = \frac{K_m}{v_{max}} \frac{1}{s_0} + \frac{1}{v_{max}}$$

This formula is known as the Lineweaver–Burk equation and shows that there is a linear relationship between $1/v_0$ and $1/s_0$.

(b) Sketch the graph of the Lineweaver–Burk equation. Use a coordinate system in which $1/s_0$ is on the horizontal axis and $1/v_0$ is on the vertical axis. Show that the resulting graph is a line that intersects the horizontal axis at $-1/K_m$ and the vertical axis at $1/v_{max}$.

(c) To determine K_m and v_{max}, we measure the initial velocity of the reaction, denoted by v_0, as a function of the concentration of the substrate, denoted by s_0, and fit a straight line through the points in a coordinate system in which the horizontal axis is $1/s_0$ and the vertical axis is $1/v_0$. Explain how to determine K_m and v_{max} from the graph.

(Note that this is an example in which a nonlogarithmic transformation is used to obtain a linear relationship. Since the reciprocals of both quantities of interest are used, the resulting plot is called a double-reciprocal plot.)

87. *(Continuation of Problem 86)* Estimating v_{max} and K_m from the Lineweaver-Burk graph as described in Problem 86 is not always satisfactory. A different transformation typically yields better estimates (Dowd and Riggs, 1965). Show that the Michaelis–Menten equation can be written as

$$\frac{v_0}{s_0} = \frac{v_{max}}{K_m} - \frac{1}{K_m} v_0$$

and explain why this transformation results in a straight line when you graph v_0 on the horizontal axis and $\frac{v_0}{s_0}$ on the vertical axis. Explain how you can estimate v_{max} and K_m from the graph.

88. *(Adapted from Reiss, 1989)* In a case study in which the maximal rates of oxygen consumption (in ml/s) of nine species of wild African mammals (Taylor et al., 1980) were plotted against body mass (in kg) on a log-log plot, it was found that the data points fell on a straight line with slope approximately equal to 0.8 and vertical-axis intercept approximately equal to 0.105. Find an equation that relates maximal oxygen consumption and body mass.

89. *(Adapted from Benton and Harper, 1997)* In vertebrates, embryos and juveniles have large heads relative to their overall body size. As the animal grows older, proportions change; for instance, the ratio of skull length to body length diminishes. That this is the case not only for living vertebrates, but also for fossil vertebrates, is shown by the following example:

Ichthyosaurs are a group of marine reptiles that appeared in the early Triassic and died out well before the end of the Cretaceous.[1] They were fish shaped and comparable in size to dolphins. In a study of 20 fossil skeletons, the following allometric relationship between skull length S (measured in cm) and backbone length B (measured in cm) was found:

$$S = 1.162 B^{0.93}$$

(a) Choose suitable transformations of S and B so that the resulting relationship is linear. Plot the transformed relationship, and find the slope and the y-intercept.

(b) Explain why the allometric equation confirms that juveniles had relatively large heads. (*Hint*: Compute the ratio of S to B for a number of different values of B—say, 10 cm, 100 cm, 500 cm—and compare.)

90. Light intensity in lakes decreases exponentially with depth. If $I(z)$ denotes the light intensity at depth z, with $z = 0$ representing the surface, then

$$I(z) = I(0)e^{-\alpha z}, \quad z \geq 0$$

where α is a positive constant called the *vertical attenuation coefficient*. Figure 1.71 shows the percentage surface radiation, defined as $100I(z)/I(0)$, as a function of depth in different lakes.

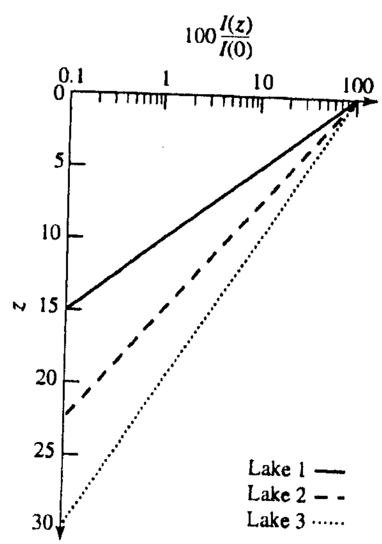

Figure 1.71 Light intensity as a function of depth for Problem 90.

(a) On the basis of the graph, estimate α for each lake.

(b) Reproduce a graph like the one in Figure 1.71 for Lake Constance (Germany) in May ($\alpha = 0.768\,\text{m}^{-1}$) and December ($\alpha = 0.219\,\text{m}^{-1}$) (data from Tilzer et al., 1982).

(c) Explain why the graphs are straight lines.

(1) The Triassic is a geological period that began about 248 million years ago and ended about 213 million years ago; the Cretaceous began about 144 million years ago and ended 65 million years ago.

91. The absorption of light in a uniform water column follows an exponential law; that is, the intensity $I(z)$ at depth z is

$$I(z) = I(0)e^{-\alpha z}$$

where $I(0)$ is the intensity at the surface (i.e., when $z = 0$) and α is the *vertical attenuation coefficient.* (We assume here that α is constant. In reality, α depends on the wavelength of the light penetrating the surface.)

(a) Suppose that 10% of the light is absorbed in the uppermost meter. Find α. What are the units of α?

(b) What percentage of the remaining intensity at 1 m is absorbed in the second meter? What percentage of the remaining intensity at 2 m is absorbed in the third meter?

(c) What percentage of the initial intensity remains at 1 m, at 2 m, and at 3 m?

(d) Plot the light intensity as a percentage of the surface intensity on both a linear plot and a log-linear plot.

(e) Relate the slope of the curve on the log-linear plot to the attenuation coefficient α.

(f) The level at which 1% of the surface intensity remains is of biological significance. Approximately, it is the level where algal growth ceases. The zone above this level is called the *euphotic zone*. Express the depth of the euphotic zone as a function of α.

(g) Compare a very clear lake with a milky glacier stream. Is the attenuation coefficient α for the clear lake greater or smaller than the attenuation coefficient α for the milky stream?

92. When plants are grown at high densities, we often observe that the number of plants decreases as plant weights increase (due to plant growth). If we plot the logarithm of the total aboveground dry-weight biomass per plant, $\log w$, against the logarithm of the density of survivors, $\log d$ (base 10), a straight line with slope $-3/2$ results. Find the equation that relates w and d, assuming that $w = 1$ g when $d = 10^3\ \text{m}^{-2}$.

In Problems 93–98, find each functional relationship on the basis of the given graph.

93. Figure 1.72
94. Figure 1.73
95. Figure 1.74
96. Figure 1.75

97. Figure 1.76 (*Hint*: This relationship is different from the ones considered so far. The x-axis is logarithmically transformed, but the y-axis is linear.)

98. Figure 1.77 (*Hint*: This relationship is different from the ones considered so far. The x-axis is logarithmically transformed, but the y-axis is linear.)

99. The free energy ΔG expended in transporting an uncharged solute across a membrane from concentration c_1 to one of concentration c_2 follows the equation

$$\Delta G = 2.303RT \log \frac{c_2}{c_1}$$

where $R = 1.99\ \text{kcal K}^{-1}\ \text{kmol}^{-1}$ is the universal gas constant and T is temperature measured in kelvins (K). Plot ΔG as a function of the concentration ratio c_2/c_1 when $T = 298$ K (25°C). Use a

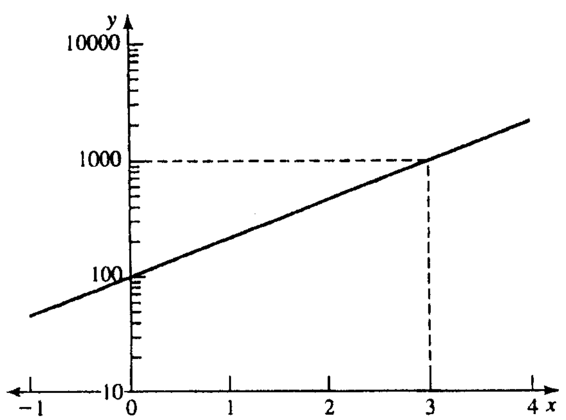

Figure 1.72 Graph for Problem 93.

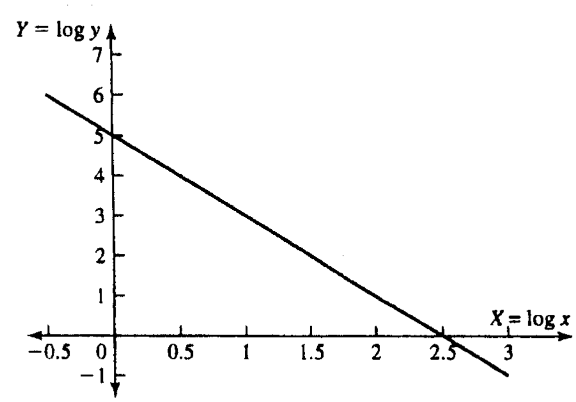

Figure 1.73 Graph for Problem 94.

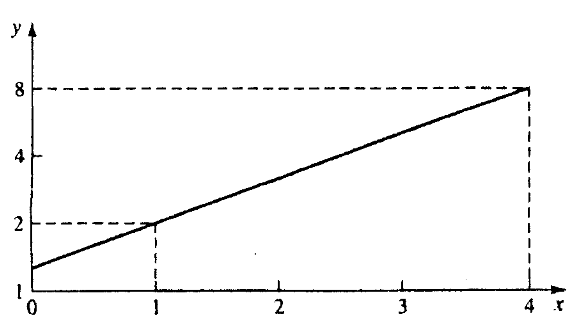

Figure 1.74 Graph for Problem 95.

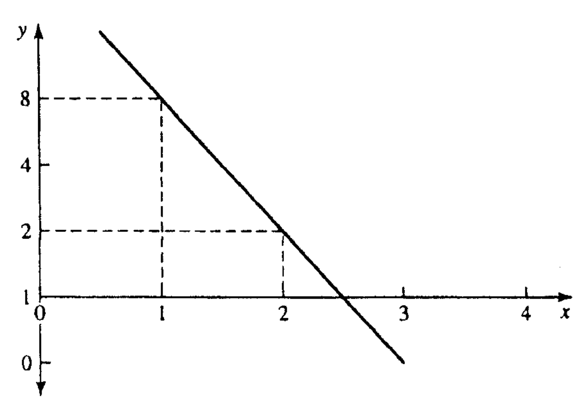

Figure 1.75 Graph for Problem 96.

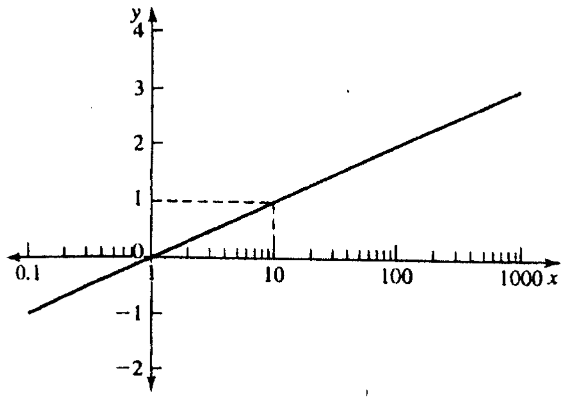

Figure 1.76 Graph for Problem 97.

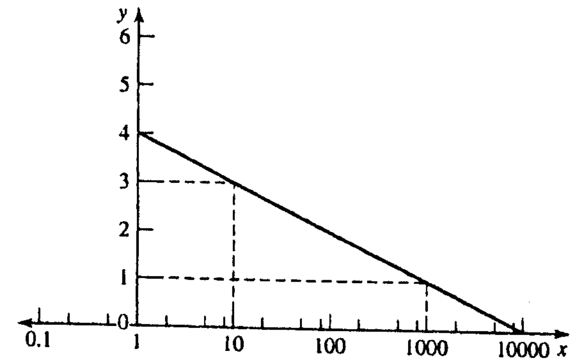

Figure 1.77 Graph for Problem 98.

coordinate system in which the vertical axis is on a linear scale and the horizontal axis is on a logarithmic scale.

100. Logistic Transformation Suppose that

$$f(x) = \frac{1}{1 + e^{-(b+mx)}} \tag{1.8}$$

A function of the form (1.8) is called a **logistic** function. The logistic function was introduced by the Dutch mathematical biologist Verhulst around 1840 to describe the growth of populations with limited food resources. Show that

$$\ln \frac{f(x)}{1 - f(x)} = b + mx \tag{1.9}$$

This transformation is called the *logistic transformation*. It is a standard transformation for linearizing functions of the form (1.8).

■ 1.3.4

101. Not every study of species richness as a function of productivity produces a hump-shaped curve. Owen (1988) studied rodent assemblages in Texas and found that the number of species was a decreasing function of productivity. Sketch a graph that would describe this situation.

102. Species diversity in a community may be controlled by disturbance frequency. The intermediate disturbance hypothesis states that species diversity is greatest at intermediate disturbance levels. Sketch a graph of species diversity as a function of disturbance level that illustrates this hypothesis.

103. Preston (1962) investigated the dependence of number of bird species on island area in the West Indian islands. He found that the number of bird species increased at a decelerating rate as island area increased. Sketch this relationship.

104. Phytoplankton converts carbon dioxide to organic compounds during photosynthesis. This process requires sunlight. It has been observed that the rate of photosynthesis is a function of light intensity: The rate of photosynthesis increases approximately linearly with light intensity at low intensities, saturates at intermediate levels, and decreases slightly at high intensities. Sketch a graph of the rate of photosynthesis as a function of light intensity.

105. Brown lemming densities in the tundra areas of North America and Eurasia show cyclic behavior: Every three to four years, lemming densities build up very rapidly, and they typically crash the next year. Sketch a graph that describes this situation.

106. Nitrogen productivity can be defined as the amount of dry matter produced per unit of nitrogen per unit of time. Experimental studies suggest that nitrogen productivity increases as a function of light intensity at a decelerating rate. Sketch a graph of nitrogen productivity as a function of light intensity.

107. A study of Borchert's (1994) investigated the relationship between stem water storage and wood density in a number of tree species in Costa Rica. The study showed that water storage is inversely related to wood density; that is, higher wood density corresponds to lower water content. Sketch a graph of water content as a function of wood density that illustrates this situation.

108. Species richness can be a hump-shaped function of productivity. In the same coordinate system, sketch two hump-shaped graphs of species richness as a function of productivity, one in which the maximum occurs at low productivity and one in which the maximum occurs at high productivity.

109. The size distribution of zooplankton in a lake is typically a hump-shaped curve; that is, if the frequency (in percent) of zooplankton is plotted against the body length of zooplankton, a curve that first increases and then decreases results. Brooks and Dodson (1965) studied the effects of introducing a planktivorous fish in a lake. They found that the composition of zooplankton after the fish was introduced shifted to smaller individuals. In the same coordinate system, sketch the size distribution of zooplankton before and after the introduction of the planktivorous fish.

110. *Daphnia* is a genus of zooplankton that comprises a number of species. The body growth rate of *Daphnia* depends on food concentration. A minimum food concentration is required for growth: Below this level, the growth rate is negative; above, it is positive. In a study by Gliwicz (1990), it was found that growth rate is an increasing function of food concentration and that the minimum food concentration required for growth decreases with increasing size of the animal. Sketch two graphs in the same coordinate system, one for a large and one for a small *Daphnia* species, that illustrates this situation.

111. Grant (1982) investigated egg weight as a function of adult body weight among 10 species of Darwin's finches. He found that the relationship between the logarithm of the average egg size and the logarithm of the average body size is linear and that smaller species lay smaller eggs and larger species lay larger eggs. Graph this relationship.

112. Grant et al. (1985) investigated the relationship between mean wing length and mean weight among males of populations of six ground finch species. They found a positive and nearly linear relationship between these two quantities. Graph this relationship.

113. Bohlen et al. (2001) investigated stream nitrate concentration along an elevation gradient at the Hubbard Brook Experimental Forest in New Hampshire. They found that the nitrate concentration in stream water declined with decreasing elevation. Sketch stream nitrate concentration as a function of elevation.

114. In Example 13, we discussed germination success as a function of temperature for varying levels of humidity. We can also consider germination success as a function of humidity for various levels of temperature. Sketch the following graphs of germination success as a function of humidity: one for low temperature, one for intermediate temperature, and one for high temperature.

115. Boulinier et al. (2001) studied the dynamics of forest bird communities. They found that the mean local extinction rate of area-sensitive species declined with mean forest patch size whereas the mean extinction rate of non-area-sensitive species did not depend on mean forest size. In the same coordinate system graph the mean extinction rate as a function of mean forest patch size for **(a)** an area-sensitive species and **(b)** a non-area-sensitive species.

116. Dalling et al. (2001) compared net photosynthetic rates of two pioneer trees—*Alseis blackiana* and *Miconia argenta*—as a function of gap size in Barro Colorado Island. They found that net photosynthetic rates (measured on a per-unit basis) increased with gap size for both trees and that the photosynthetic rate for *Miconia argenta* was higher than that for *Alseis blackiana*. In the same coordinate system, graph the net photosynthetic rates as functions of gap size for both tree species.

Chapter 1 Key Terms

Discuss the following definitions and concepts:

1. Real numbers
2. Intervals: open, closed, half-open
3. Absolute value
4. Proportional
5. Lines: standard form, point–slope form, slope–intercept form
6. Parallel and perpendicular lines
7. Circle: radius, center, equation of circle, unit circle
8. Angle: radians, degrees
9. Trigonometric identities
10. Complex numbers: real part, imaginary part
11. Function: domain, codomain, range, image
12. Symmetry of functions: even, odd
13. Composition of functions
14. Polynomial
15. Degree of a polynomial
16. Chemical reaction: law of mass action
17. Rational function
18. Growth rate
19. Specific growth rate and per capita growth rate
20. Monod growth function
21. Power function
22. Allometry and scaling relations
23. Exponential function
24. Exponential growth
25. Natural exponential base
26. Radioactive decay
27. Half-life
28. Inverse function, one to one
29. Logarithmic function
30. Relationship between exponential and logarithmic functions
31. Periodic function
32. Trigonometric function
33. Amplitude, period
34. Translation: horizontal, vertical
35. Logarithmic scale
36. Order of magnitude
37. Logarithmic transformation
38. Log-log plot
39. Semilog plot

Chapter 1 Review Problems

1. Population Growth Suppose that the number of bacteria in a petri dish is given by

$$B(t) = 10,000e^{0.1t}$$

where t is measured in hours.

(a) How many bacteria are present at $t = 0, 1, 2, 3$, and 4?

(b) Find the time t when the number of bacteria reaches 100,000.

2. Population Decline Suppose that a pathogen is introduced into a population of bacteria at time 0. The number of bacteria then declines as

$$B(t) = 25,000e^{-2t}$$

where t is measured in hours.

(a) How many bacteria are left after 3 hours?

(b) How long will it be until only 1% of the initial number of bacteria are left?

3. Chemical Reaction Consider the chemical reaction

$$A + 2B \rightarrow AB_2$$

Assume that the reaction occurs in a closed vessel and that the initial concentrations of A and B are $a = [A]$ and $b = [B]$, respectively.

(a) Explain why the reaction rate $R(x)$ is given by

$$R(x) = k(a - x)(b - 2x)^2$$

where $x = [AB_2]$.

(b) Show that $R(x)$ is a polynomial and determine its degree.

(c) Graph $R(x)$ for the relevant values of x when $a = 5$, $b = 6$, and $k = 0.3$.

4. History of Mathematics Euclid, a Greek mathematician who lived around 300 B.C., wrote the *Elements*, by far the most important mathematical text of that period. The book, arranged in 13 volumes, is a systematic exposition of most of the mathematical knowledge of that time. In Book III, Euclid discusses the construction of a tangent to a circle at a point P on the circle. To phrase the construction in modern terminology, we draw a straight line through the point P that is perpendicular to the line through the center of the circle and the point P on the circle.

(a) Use this geometric construction to find the equation of the line that is tangent to the unit circle at the point $(\frac{1}{2}\sqrt{3}, \frac{1}{2})$.

(b) Determine the angle θ between the positive x-axis and the tangent line found in (a). What is the relationship between the angle θ and the slope of the tangent line found in (a)?

5. Hypothetical Plants To compare logarithmic and exponential growth, we consider two hypothetical plants that are of the same genus, but that exhibit rather different growth rates. Both plants produce a single leaf whose length continues to increase as long as the plant is alive. One plant is called *Growthus logarithmiensis*; the other one is called *Growthus exponentialis*. The length L (measured in feet) of the leaf of *G. logarithmiensis* at age t (measured in years) is given by

$$L(t) = \ln(t+1), \quad t \geq 0$$

The length E (measured in feet) of the leaf of *G. exponentialis* at age t (measured in years), is given by

$$E(t) = e^t - 1, \quad t \geq 0$$

(a) Find the length of each leaf after 1, 10, 100, and 1000 years.
(b) How long would it take for the leaf of *G. exponentialis* to reach a length of 233,810 mi, the average distance from the earth to the moon? (Note that 1 mi = 5280 ft.) How long would the leaf of *G. logarithmiensis* then be?
(c) How many years would it take the leaf of *G. logarithmiensis* to reach a length of 233,810 mi? Compare this with the length of time since life appeared on earth, about 3500 million years. If *G. logarithmiensis* had appeared 3,500 million years ago, and if there was a plant of this species that had actually survived throughout the entire period, how long would its leaf be today?
(d) Plants started to conquer land only in the late Ordovician period, around 450 million years ago.[2] If both *G. exponentialis* and *G. logarithmiensis* had appeared then, and there was a plant of each species that had actually survived throughout the entire period, how long would their respective leaves be today?

6. Population Growth In Chapter 3 of *The Origin of Species* (Darwin, 1859), Charles Darwin asserts that a "struggle for existence inevitably follows from the high rate at which all organic beings tend to increase. ... Although some species may be now increasing, more or less rapidly, in numbers, all cannot do so, for the world would not hold them." To illustrate this point, he continues as follows:

> There is no exception to the rule that every organic being naturally increases at so high a rate, that, if not destroyed, the earth would soon be covered by the progeny of a single pair. Even slow-breeding man has doubled in twenty-five years, and at this rate, in a few thousand years, there would literally not be standing room for his progeny.

Starting with a single pair, compute the world's population after 1000 years and after 2000 years under Darwin's assumption that the world's population doubles every 25 years, and find the resulting population densities (number of people per square foot). To answer the last part, you need to know that the earth's diameter is about 7900 mi, the surface of a sphere is $4\pi r^2$, where r is the radius of a sphere, and the continents make up about 29% of the earth's surface. (Note that 1 mi = 5280 ft.)

7. Population Growth Assume that a population grows q% each year. How many years will it take the population to double in size? Give the functional relationship between the doubling time T and the annual percentage increase q. Produce a table that shows the doubling time T as a function of q for $q = 1, 2, \ldots, 10$, and graph T as a function of q. What happens to T as q gets closer to 0?

(2) The Ordovician lasted from about 505 million years ago to about 438 million years ago.

8. Beverton–Holt Recruitment Curve Many organisms show density-dependent mortality. The following is a simple mathematical model that incorporates this effect: Denote the density of parents by N_b and the density of *surviving* offspring by N_a.
(a) Suppose that without density-dependent mortality, the number of surviving offspring per parent is equal to R. Show that if we plot N_b/N_a versus N_b, the result is a horizontal line with y-intercept $1/R$. That is,

$$\frac{N_b}{N_a} = \frac{1}{R}$$

or

$$N_a = R \cdot N_b$$

The constant R is called the **net reproductive rate.**
(b) To include density-dependent mortality, we assume that N_b/N_a is an increasing function of N_b. The simplest way to do this is to assume that the graph of N_b/N_a versus N_b is a straight line with y-intercept $1/R$ and that goes through the point $(K, 1)$. Show that this implies that

$$N_a = \frac{RN_b}{1 + \frac{(R-1)N_b}{K}}$$

This relationship is called the Beverton–Holt recruitment curve.
(c) Explain in words why, for small initial densities N_b, the model described by the Beverton–Holt recruitment curve behaves like the model for density-independent mortality described in (a).
(d) Show that if $N_b = K$, then $N_a = K$. Furthermore, show that $N_b < K$ implies $N_b < N_a < K$ and that $N_b > K$ implies $K < N_a < N_b$. Explain in words what this means. (Note that K is called the *carrying capacity*.)
(e) Plot N_a as a function of N_b for $R = 2$ and $K = 20$. What happens for large values of N_b? Explain in words what this means.

9. Fish Yield *(Adapted from Moss, 1980)* Oglesby (1977) investigated the relationship between annual fish yield (Y) and summer phytoplankton chlorophyll concentration (C). Fish yield was measured in grams dry weight per square meter per year, and the chlorophyll concentration was measured in micrograms per liter. Data from 19 lakes, mostly in the Northern Hemisphere, yielded the following relationship:

$$\log_{10} Y = 1.17 \log_{10} C - 1.92 \qquad (1.10)$$

(a) Plot $\log_{10} Y$ as a function of $\log_{10} C$.
(b) Find the relationship between Y and C; that is, write Y as a function of C. Explain the advantage of the log–log transformation resulting in (1.10) versus writing Y as a function of C. [*Hint*: Try to plot Y as a function of C, and compare with your answer in (a).]
(c) Find the predicted yield (Y_p) as a function of the current yield (Y_c) if the current summer phytoplankton chlorophyll concentration were to double.
(d) By what percentage would the summer phytoplankton chlorophyll concentration need to increase to obtain a 10% increase in fish yield?

10. Radioactive Decay *(Adapted from Moss, 1980)* To trace the history of a lake, a sample of mud from a core is taken and dated. One dating method uses radioactive isotopes. The C^{14} method is effective for sediments that are younger than 60,000 years. The $C^{14} : C^{12}$ ratio has been essentially constant in the atmosphere over a long time, and living organisms take up carbon in that ratio.

Upon death, the uptake of carbon ceases and C^{14} decays, which changes the $C^{14}:C^{12}$ ratio according to

$$\left(\frac{C^{14}}{C^{12}}\right)_t = \left(\frac{C^{14}}{C^{12}}\right)_{\text{initial}} e^{-\lambda t}$$

where t is the time since death.

(a) If the $C^{14}:C^{12}$ ratio in the atmosphere is 10^{-12} and the half-life of C^{14} is 5730 years, find an expression for t, the age of the material being dated, as a function of the $C^{14}:C^{12}$ ratio in the material being dated.

(b) Use your answer in (a) to find the age of a mud sample from a core for which the $C^{14}:C^{12}$ ratio is 1.61×10^{-13}.

11. Fossil Coral Growth *(Adapted from Futuyama, 1995, and Dott and Batten, 1976)* Corals deposit a single layer of lime each day. In addition, seasonal fluctuation in the thickness of the layers allows for grouping them into years. In modern corals, we can count 365 layers per year. J. Wells, a paleontologist, counted such growth layers on fossil corals. To his astonishment, he found that Devonian[3] corals that lived about 380 million years ago had about 400 daily layers per year.

(a) Today, the earth rotates about its axis every 24 hours and revolves around the sun every $365\frac{1}{4}$ days. Astronomers have determined that the earth's rotation has slowed down in recent centuries at the rate of about 2 seconds every 100,000 years. That is, 100,000 years ago, a day was 2 seconds shorter than today. Extrapolate the slowdown back to the Devonian, and determine the length of a day and the length of a year back when Wells's corals lived. (*Hint:* The number of hours per year remains constant. Why?)

(b) Find a linear equation that relates geologic time (in million of years) to the number of hours per day at a given time.

(c) Algal stromatolites also show daily layers. A sample of some fossil stromatolites showed 400 to 420 daily layers per year. Use your answer in (b) to date the stromatolites.

12. Tree Growth The height y in feet of a certain tree as a function of age x in years can be approximated by

$$y = 132e^{-20/x}$$

(a) Use a graphing calculator to plot the graph of this function. Describe in words how the tree grows, paying particular attention to questions such as the following: Does the tree grow equally fast over time? What happens when the tree is young? What happens when the tree is old?

(b) How many years will it take for the tree to reach 100 ft in height?

(c) Can the tree ever reach a height of 200 ft? Is there a final height—that is, a maximum height that the tree will eventually reach?

13. Model for Aging The probability that an individual lives beyond age t is called the *survivorship function* and is denoted by $S(t)$. The Weibull model is a popular model in reliability theory and in studies of biological aging. Its survivorship function is described by two parameters, λ and β, and is given by

$$S(t) = \exp[-(\lambda t)^{\beta}]$$

Mortality data from a *Drosophila melanogaster* population in Dr. Jim Curtsinger's lab at the University of Minnesota were collected and fitted to this model separately for males and females (Pletcher, 1998). The following parameter values were obtained (t was measured in days):

Sex	λ	β
Males	0.019	3.41
Females	0.022	3.24

(a) Use a graphing calculator to sketch the survivorship function for both the female and male populations.

(b) For each population, find the value of t for which the probability of living beyond that age is 1/2.

(c) If you had a male and a female of this species, which would you expect to live longer?

14. Carbon Isotope Carbon has two stable isotopes: C^{12} and C^{13}. Organic material contains both stable isotopes but the ratio $[C^{13}]:[C^{12}]$ in organic material is smaller than that in inorganic material, reflecting the fact that light carbon (C^{12}) is preferentially taken up by plants during photosynthesis. This process is called *isotopic fractionation* and is measured as

$$\delta^{13}C = \left[\frac{([C^{13}]:[C^{12}])_{\text{sample}}}{([C^{13}]:[C^{12}])_{\text{standard}}} - 1\right]$$

The standard is taken from the isotope ratio in the carbon of belemnite shells found in the Cretaceous Pedee formation of South Carolina. Explain, on the basis of the preceding information, why the following quotation from Krauskopf and Bird (1995) makes sense:

> The low [negative] values of $\delta^{13}C$ in the hydrocarbons of petroleum are one of the important bits of evidence for ascribing the origin of petroleum to the alteration of organic material rather than to condensation of primeval gases from the Earth's interior.

15. Chemical Reaction The speed of an enzymatic reaction is frequently described by the Michaelis–Menten equation

$$v = \frac{ax}{k + x}$$

where v is the velocity of the reaction, x is the concentration of the substrate, a is the maximum reaction velocity, and k is the substrate concentration at which the velocity is half of the maximum velocity. This curve describes how the reaction velocity depends on the substrate concentration.

(a) Show that when $x = k$, the velocity of the reaction is half the maximum velocity.

(b) Show that an 81-fold change in substrate concentration is needed to change the velocity from 10% to 90% of the maximum velocity, regardless of the value of k.

16. Lake Acidification Atmospheric pollutants can cause acidification of lakes (by acid rain). This can be a serious problem for lake organisms; for instance, in fish the ability of hemoglobin to transport oxygen decreases with decreasing pH levels of the water. Experiments with the zooplankton *Daphnia magna* showed a negligible decline in survivorship at pH = 6, but a marked decline in survivorship at pH = 3.5, resulting in no survivors after just eight hours. Illustrate graphically the percentage survivorship as a function of time for pH = 6 and pH = 3.5.

(3) The Devonian period lasted from about 408 million years ago to about 360 million years ago.

17. Lake Chemistry The pH level of a lake controls the concentrations of harmless ammonium ions (NH_4^+) and toxic ammonia (NH_3) in the lake. For pH levels below 8, concentrations of NH_4^+ ions are little affected by changes in the pH value, but they decline over many orders of magnitude as pH levels increase beyond pH = 8. By contrast, NH_3 concentrations are negligible at low pH, increase over many orders of magnitude as the pH level increases, and reach a high plateau at about pH = 10 (after which levels of NH_3 are little affected by changes in pH levels). Illustrate the behavior of $[NH_4^+]$ and $[NH_3]$ graphically.

18. Development and Growth Egg development times of the zooplankton *Daphnia longispina* depend on temperature. It takes only about 3 days at 20°C, but almost 20 days at 5°C, for an egg to develop and hatch. When graphed on a log–log plot, egg development time (in days) as a function of temperature (in °C) is a straight line.

(a) Sketch a graph of egg development time as a function of temperature on a log-log plot.

(b) Use the data to find the function that relates egg development time and temperature for *D. longispina*.

(c) Use your answer in (b) to predict egg development time of *D. longispina* at 10°C.

(d) Suppose you measured egg development time in hours and temperature in Fahrenheit. Would you still find a straight line on a log-log plot?

19. Resource Model Organisms consume resources. The rate of resource consumption, denoted by v, depends on resource concentration, denoted by S. The *Blackman* model of resource consumption assumes a linear relationship between resource consumption rate and resource concentration: Below a threshold concentration (S_k), the consumption rate increases linearly with $S = 0$ when $v = 0$; when $S = S_k$, the consumption rate v reaches its maximum value v_{max}; for $S > S_k$, the resource consumption rate stays at the maximum value v_{max}. A function like this, with a sharp transition, cannot be described analytically by just one expression; it needs to be defined piecewise:

$$v = \begin{cases} g(S) & \text{for } 0 \leq S < S_k \\ v_{max} & \text{for } S \geq S_k \end{cases}$$

Find $g(S)$, and graph the resource consumption rate v as a function of resource concentration S.

20. Light Intensity Light intensity in lakes decreases exponentially with depth. If $I(z)$ denotes the light intensity at depth z, with $z = 0$ representing the surface, then

$$I(z) = I(0)e^{-\alpha z}, \quad z \geq 0$$

where α is a positive constant called the *vertical attenuation coefficient*. This coefficient depends on the wavelength of the light and on the amount of dissolved matter and particles in the water. In the following, we assume that the water is pure:

(a) About 65% of red light (720 nm) is absorbed in the first meter. Find α.

(b) About 5% of blue light (475 nm) is absorbed in the first meter. Find α.

(c) Explain in words why a diver would not see red hues a few meters below the surface of a lake.

21. Light Intensity Light intensity in lakes decreases with depth according to the relationship

$$I(z) = I(0)e^{-\alpha z}, \quad z \geq 0$$

where $I(z)$ denotes the light intensity at depth z, $z = 0$ represents the surface, and α is a positive constant denoting the vertical attenuation coefficient. The depth where light intensity is about 1% of the surface light intensity is important for photosynthesis in phytoplankton: Below this level, photosynthesis is insufficient to compensate for respiratory losses. The 1% level is called the *compensation level.* An often used and relatively reliable method for determining the compensation level is the *Secchi disk* method. A Secchi disk is a white disk with radius 10 cm. The disk depth is the depth at which the disk disappears from the viewer. Twice this depth approximately coincides with the compensation level.

(a) Find α for a lake with Secchi disk depth of 9 m.

(b) Find the Secchi disk depth for a lake with $\alpha = 0.473\ m^{-1}$.

22. Population Growth Assume that the population size $N(t)$ at time $t \geq 0$ is given by

$$N(t) = N_0 e^{rt}$$

with $N_0 = N(0)$. The parameter r is called the *average annual growth rate.*

(a) Show that

$$r = \ln \frac{N(t+1)}{N(t)} \tag{1.11}$$

Formula (1.11) is used, for instance, by the U.S. Census Bureau to track world population growth.

(b) Suppose a population doubles in size within a single year.

(i) What is the percent increase of the population during that year?

(ii) What is the average annual growth rate in percent during that year, according to (1.11)?

(c) Suppose the average annual growth rate of a population is 1.3%. How many years will it take the population to double in size?

(d) To calculate the doubling time of a growing population with a constant average annual growth rate, we divide the percent average annual growth rate into 70. Apply this "Rule of 70" to (c) and compare your answers. Derive the "Rule of 70."