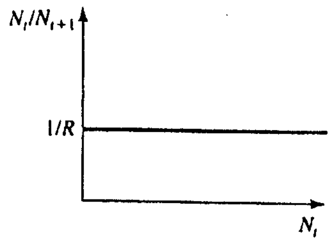

Figure 2.7 The graph of the parent–offspring ratio $\frac{N_t}{N_{t+1}}$ as a function of N_t when $N_t > 0$.

offspring, regardless of the current population density. Such growth is called **density independent.**

When $R > 1$, it follows that $1/R$, the parent–offspring ratio, is less than 1, implying that the number of offspring exceeds the number of parents. Density-independent growth with $R > 1$ results in an ever-increasing population size. This model eventually becomes biologically unrealistic, since any population will sooner or later experience food or habitat limitations that will limit its growth. (We will discuss models that include such limitations in Section 2.3.)

The density independence in exponential growth is reflected in a graph of N_t/N_{t+1} as a function of N_t, which is a horizontal line at level $1/R$ (Figure 2.7).

As before, only a selected number of points are realized on the graph of N_t/N_{t+1} as a function of N_t, and time is implicit in the graph. (See Figure 2.8, with $R = 2$ and $N_0 = 1$.)

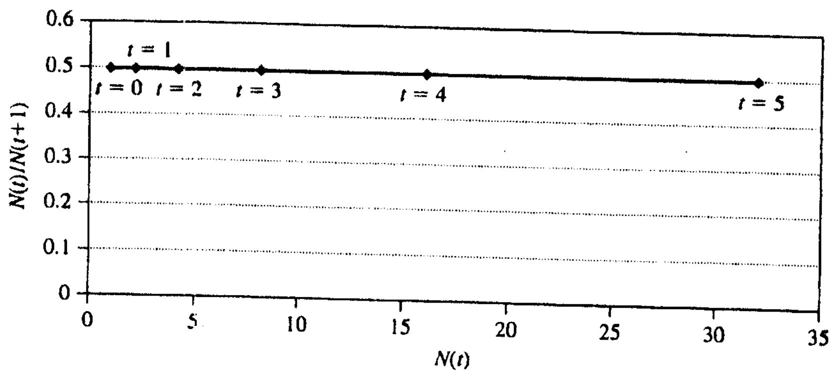

Figure 2.8 The graph of the parent–offspring ratio $\frac{N_t}{N_{t+1}}$ as a function of N_t when $N_t = 1$ and $R = 2$.

Section 2.1 Problems

In Problems 1–4, produce a table for $t = 0, 1, 2, \ldots, 5$ and graph the function N_t.

1. $N_t = 3^t$
2. $N_t = 10 \cdot 2^t$
3. $N_t = \frac{25}{4^t}$
4. $N_t = (0.3)(0.9)^t$

In Problems 5–10, give a formula for $N(t)$, $t = 0, 1, 2, \ldots,$ on the basis of the information provided.

5. $N_0 = 2$; population doubles every 20 minutes; one unit of time is 20 minutes
6. $N_0 = 4$; population doubles every 40 minutes; one unit of time is 40 minutes
7. $N_0 = 1$; population doubles every 40 minutes; one unit of time is 80 minutes
8. $N_0 = 6$; population doubles every 40 minutes; one unit of time is 60 minutes
9. $N_0 = 2$; population quadruples every 30 minutes; one unit of time is 15 minutes
10. $N_0 = 10$; population quadruples every 20 minutes; one unit of time is 10 minutes
11. Suppose $N_t = 20 \cdot 4^t$, $t = 0, 1, 2, \ldots,$ and one unit of time corresponds to 3 hours. Determine the amount of time it takes the population to double in size.
12. Suppose $N_t = 100 \cdot 2^t$, $t = 0, 1, 2, \ldots,$ and one unit of time corresponds to 2 hours. Determine the amount of time it takes the population to triple in size.
13. A strain of bacteria reproduces asexually every hour. That is, every hour, each bacterial cell splits into two cells. If, initially, there is one bacterium, find the number of bacterial cells after 1 hour, 2 hours, 3 hours, 4 hours, and 5 hours.
14. A strain of bacteria reproduces asexually every 30 minutes. That is, every 30 minutes, each bacterial cell splits into two cells. If, initially, there is one bacterium, find the number of bacterial cells after 1 hour, 2 hours, 3 hours, 4 hours, and 5 hours.
15. A strain of bacteria reproduces asexually every 23 minutes. That is, every 23 minutes, each bacterial cell splits into two cells. If, initially, there is 1 bacterium, how long will it take until there are 128 bacteria?
16. A strain of bacteria reproduces asexually every 42 minutes. That is, every 42 minutes, each bacterial cell splits into two cells. If, initially, there is 1 bacterium, how long will it take until there are 512 bacteria?
17. A strain of bacteria reproduces asexually every 10 minutes. That is, every 10 minutes, each bacterial cell splits into two cells. If, initially, there are 3 bacteria, how long will it take until there are 96 bacteria?
18. A strain of bacteria reproduces asexually every 50 minutes. That is, every 50 minutes, each bacterial cell splits into two cells. If, initially, there are 10 bacteria, how long will it take until there are 640 bacteria?

19. Find the exponential growth equation for a population that doubles in size every unit of time and that has 40 individuals at time 0.

20. Find the exponential growth equation for a population that doubles in size every unit of time and that has 53 individuals at time 0.

21. Find the exponential growth equation for a population that triples in size every unit of time and that has 20 individuals at time 0.

22. Find the exponential growth equation for a population that triples in size every unit of time and that has 72 individuals at time 0.

23. Find the exponential growth equation for a population that quadruples in size every unit of time and that has five individuals at time 0.

24. Find the exponential growth equation for a population that quadruples in size every unit of time and that has 17 individuals at time 0.

25. Find the recursion for a population that doubles in size every unit of time and that has 20 individuals at time 0.

26. Find the recursion for a population that doubles in size every unit of time and that has 37 individuals at time 0.

27. Find the recursion for a population that triples in size every unit of time and that has 10 individuals at time 0.

28. Find the recursion for a population that triples in size every unit of time and that has 84 individuals at time 0.

29. Find the recursion for a population that quadruples in size every unit of time and that has 30 individuals at time 0.

30. Find the recursion for a population that quadruples in size every unit of time and that has 62 individuals at time 0.

In Problems 31–34, graph the functions $f(x) = a^x$, $x \in [0, \infty)$, and $N_t = R^t$, $t \in \mathbf{N}$, together in one coordinate system for the indicated values of a and R.

31. $a = R = 2$ **32.** $a = R = 3$

33. $a = R = 1/2$ **34.** $a = R = 1/3$

In Problems 35–46, find the population sizes for $t = 0, 1, 2, \ldots, 5$ for each recursion.

35. $N_{t+1} = 2N_t$ with $N_0 = 3$ **36.** $N_{t+1} = 2N_t$ with $N_0 = 5$

37. $N_{t+1} = 3N_t$ with $N_0 = 2$ **38.** $N_{t+1} = 3N_t$ with $N_0 = 7$

39. $N_{t+1} = 5N_t$ with $N_0 = 1$ **40.** $N_{t+1} = 7N_t$ with $N_0 = 4$

41. $N_{t+1} = \frac{1}{2}N_t$ with $N_0 = 1024$

42. $N_{t+1} = \frac{1}{2}N_t$ with $N_0 = 4096$

43. $N_{t+1} = \frac{1}{3}N_t$ with $N_0 = 729$

44. $N_{t+1} = \frac{1}{3}N_t$ with $N_0 = 3645$

45. $N_{t+1} = \frac{1}{5}N_t$ with $N_0 = 31250$

46. $N_{t+1} = \frac{1}{4}N_t$ with $N_0 = 8192$

In Problems 47–58, write N_t as a function of t for each recursion.

47. $N_{t+1} = 2N_t$ with $N_0 = 15$ **48.** $N_{t+1} = 2N_t$ with $N_0 = 7$

49. $N_{t+1} = 3N_t$ with $N_0 = 12$ **50.** $N_{t+1} = 3N_t$ with $N_0 = 3$

51. $N_{t+1} = 4N_t$ with $N_0 = 24$ **52.** $N_{t+1} = 5N_t$ with $N_0 = 17$

53. $N_{t+1} = \frac{1}{2}N_t$ with $N_0 = 5000$

54. $N_{t+1} = \frac{1}{2}N_t$ with $N_0 = 2300$

55. $N_{t+1} = \frac{1}{3}N_t$ with $N_0 = 8000$

56. $N_{t+1} = \frac{1}{3}N_t$ with $N_0 = 3500$

57. $N_{t+1} = \frac{1}{5}N_t$ with $N_0 = 1200$

58. $N_{t+1} = \frac{1}{7}N_t$ with $N_0 = 6400$

In Problems 59–66, graph the line $N_{t+1} = RN_t$ in the N_t–N_{t+1} plane for the indicated value of R and locate the points (N_t, N_{t+1}), $t = 0$, 1, and 2, for the given value of N_0.

59. $R = 2$, $N_0 = 2$ **60.** $R = 2$, $N_0 = 3$

61. $R = 3$, $N_0 = 1$ **62.** $R = 4$, $N_0 = 2$

63. $R = \frac{1}{2}$, $N_0 = 16$ **64.** $R = \frac{1}{2}$, $N_0 = 64$

65. $R = \frac{1}{3}$, $N_0 = 81$ **66.** $R = \frac{1}{4}$, $N_0 = 16$

In Problems 67–74, graph the line $\frac{N_t}{N_{t+1}} = \frac{1}{R}$ in the N_t–$\frac{N_t}{N_{t+1}}$ plane for the indicated value of R and locate the points $(N_t, \frac{N_t}{N_{t+1}})$, $t = 0$, 1, 2, for the given value of N_0. Find the parent–offspring ratio.

67. $R = 2$, $N_0 = 2$ **68.** $R = 2$, $N_0 = 4$

69. $R = 3$, $N_0 = 2$ **70.** $R = 4$, $N_0 = 1$

71. $R = \frac{1}{2}$, $N_0 = 16$ **72.** $R = \frac{1}{2}$, $N_0 = 128$

73. $R = \frac{1}{3}$, $N_0 = 27$ **74.** $R = \frac{1}{4}$, $N_0 = 64$

75. A bird population lives in a habitat where the number of nesting sites is a limiting factor in population growth. In which of the following cases would you expect that the growth of this bird population over the next few generations could be reasonably well approximated by exponential growth?

(a) All nesting sites are occupied.

(b) The bird population just invaded the habitat, and the population size is still much smaller than the available nesting sites.

(c) In the previous year, a hurricane killed more than 90% of the birds in this habitat.

76. Pollen records show that the number of Scotch pine (*Pinus sylvestris*) grew exponentially for about 500 years after colonization of the Norfolk region of Great Britain about 9500 years ago. Can you find a possible explanation for this growth?

77. Exponential growth generally occurs when population growth is density independent. List conditions under which a population might stop growing exponentially.

2.2 Sequences

2.2.1 What Are Sequences?

Before we explore other discrete-time population models, we need to develop further the theory of functions with domain **N**. The functions are of the form

$$\begin{aligned} f : \mathbf{N} &\to \mathbf{R} \\ n &\to f(n) \end{aligned}$$

Thus, a_3 is the same as a_1, a_4 is the same as a_2 and hence a_0, and so on.

The last two examples illustrate that fixed points are only *candidates* for limi and that, depending on the initial condition, the sequence $\{a_n\}$ may or may not co verge to a given fixed point. If we know, however, that a sequence $\{a_n\}$ does converg then the limit of the sequence must be one of the fixed points.

There is a graphical method for finding fixed points, which we will mention brief here: If the recursion is of the form $a_{n+1} = g(a_n)$, then a fixed point satisfies $a = g(a$ This suggests that if we graph $y = g(x)$ and $y = x$ in the same coordinate systen then fixed points are located where the two graphs intersect, as shown in Figure 2.1

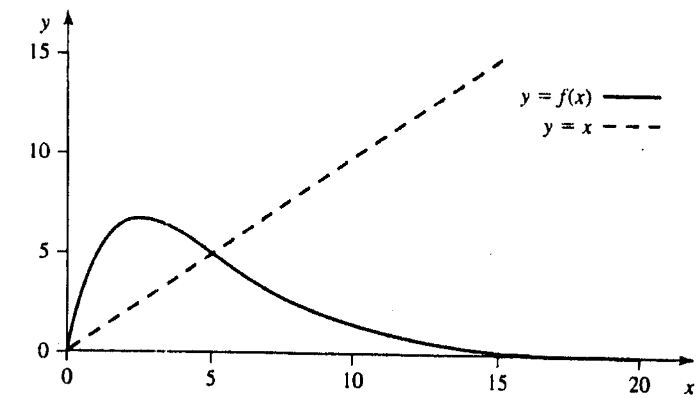

Figure 2.13 A graphical way to find fixed points. (See text for explanation.)

We will return to the relationship between fixed points and limits in Sectic 5.6, where we will learn methods that allow us to determine whether a sequenc converges to a particular fixed point.

Section 2.2 Problems

■ 2.2.1

In Problems 1–16, determine the values of the sequence $\{a_n\}$ for $n = 0, 1, 2, \ldots, 5$.

1. $a_n = n$

2. $a_n = 3n^2$

3. $a_n = \dfrac{1}{n+2}$

4. $f(n) = \dfrac{1}{1+n^2}$

5. $f(n) = \dfrac{1}{(1+n)^2}$

6. $a_n = \dfrac{1}{\sqrt{n+1}}$

7. $f(n) = (n+1)^2$

8. $f(n) = \sqrt{n+4}$

9. $a_n = (-1)^n n$

10. $a_n = \dfrac{(-1)^n}{(n+1)^2}$

11. $a_n = \dfrac{n^2}{n+1}$

12. $a_n = n^3\sqrt{n+1}$

13. $f(n) = e^{\sqrt{n}}$

14. $f(n) = 3e^{-0.1n}$

15. $f(n) = \left(\dfrac{1}{3}\right)^n$

16. $f(n) = 2^{0.2n}$

In Problems 17–24, find the next four values of the sequence $\{a_n\}$ on the basis of the values of $a_0, a_1, a_2, \ldots, a_5$.

17. $1, 2, 3, 4, 5$

18. $0, 1, \sqrt{2}, \sqrt{3}, \sqrt{4}$

19. $1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \dfrac{1}{25}$

20. $-1, \dfrac{1}{4}, -\dfrac{1}{9}, \dfrac{1}{16}, -\dfrac{1}{25}$

21. $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}$

22. $\dfrac{1}{5}, \dfrac{4}{10}, \dfrac{9}{17}, \dfrac{16}{26}, \dfrac{25}{37}$

23. $\sqrt{1+e}, \sqrt{2+e^2}, \sqrt{3+e^3}, \sqrt{4+e^4}, \sqrt{5+e^5}$

24. $\sin\dfrac{\pi}{2}, -\sin\dfrac{\pi}{4}, \sin\dfrac{\pi}{6}, -\sin\dfrac{\pi}{8}, \sin\dfrac{\pi}{10}$

In Problems 25–36, find an expression for a_n on the basis of t values of $a_0, a_1, a_2, \ldots$.

25. $0, 1, 2, 3, 4, \ldots$

26. $0, 2, 4, 6, 8, \ldots$

27. $1, 2, 4, 8, 16, \ldots$

28. $1, 3, 5, 7, 9, \ldots$

29. $1, \dfrac{1}{3}, \dfrac{1}{9}, \dfrac{1}{27}, \dfrac{1}{81}, \ldots$

30. $\dfrac{1}{3}, \dfrac{2}{5}, \dfrac{3}{7}, \dfrac{4}{9}, \dfrac{5}{11}, \ldots$

31. $-1, 2, -3, 4, -5, \ldots$

32. $2, -4, 6, -8, 10, \ldots$

33. $-\dfrac{1}{2}, \dfrac{1}{3}, -\dfrac{1}{4}, \dfrac{1}{5}, -\dfrac{1}{6}, \ldots$

34. $\dfrac{1}{2}, -\dfrac{1}{8}, \dfrac{1}{18}, -\dfrac{1}{32}, \dfrac{1}{50}, \ldots$

35. $\sin(\pi), \sin(2\pi), \sin(3\pi), \sin(4\pi), \sin(5\pi), \ldots$

36. $-\cos\dfrac{\pi}{2}, \cos\dfrac{\pi}{4}, -\cos\dfrac{\pi}{6}, \cos\dfrac{\pi}{8}, -\cos\dfrac{\pi}{10}, \ldots$

■ 2.2.2

In Problems 37–44, write the first five terms of the sequence $\{a$, $n = 0, 1, 2, 3, \ldots$, and find $\lim_{n\to\infty} a_n$.

37. $a_n = \dfrac{1}{n+2}$

38. $a_n = \dfrac{2}{n+1}$

39. $a_n = \dfrac{n}{n+1}$

40. $a_n = \dfrac{2n}{n+2}$

41. $a_n = \dfrac{1}{n^2+1}$

42. $a_n = \dfrac{1}{\sqrt{n+1}}$

43. $a_n = \frac{(-1)^n}{n+1}$

44. $a_n = \frac{(-1)^n}{n^3+3}$

In Problems 45–52, write the first five terms of the sequence $\{a_n\}$, $n = 0, 1, 2, 3, \ldots$, and determine whether $\lim_{n\to\infty} a_n$ exists. If the limit exists, find it.

45. $a_n = \frac{n^2}{n+1}$

46. $a_n = \frac{n^3}{n+1}$

47. $a_n = \sqrt{n}$

48. $a_n = n^2$

49. $a_n = 2^n$

50. $a_n = \left(\frac{1}{2}\right)^n$

51. $a_n = 3^n$

52. $a_n = \left(\frac{1}{3}\right)^n$

Formal Definition of Limits: In Problems 53–64, $\lim_{n\to\infty} a_n = a$. Find the limit a, and determine N so that $|a_n - a| < \epsilon$ for all $n > N$ for the given value of ϵ.

53. $a_n = \frac{1}{n}, \epsilon = 0.01$

54. $a_n = \frac{1}{n}, \epsilon = 0.02$

55. $a_n = \frac{1}{n^2}, \epsilon = 0.01$

56. $a_n = \frac{1}{n^2}, \epsilon = 0.001$

57. $a_n = \frac{1}{\sqrt{n}}, \epsilon = 0.1$

58. $a_n = \frac{1}{\sqrt{n}}, \epsilon = 0.05$

59. $a_n = \frac{(-1)^n}{n}, \epsilon = 0.01$

60. $a_n = \frac{(-1)^n}{n}, \epsilon = .001$

61. $a_n = \frac{n}{n+1}, \epsilon = 0.01$

62. $a_n = \frac{n+1}{n}, \epsilon = .05$

63. $a_n = \frac{n^2}{n^2+1}, \epsilon = 0.01$

64. $a_n = \frac{n^2}{n^2+1}, \epsilon = .001$

Formal Definition of Limits: In Problems 65–70, use the formal definition of limits to show that $\lim_{n\to\infty} a_n = a$; that is, find N such that for every $\epsilon > 0$, there exists an N such that $|a_n - a| < \epsilon$ whenever $n > N$.

65. $\lim_{n\to\infty} \frac{1}{n} = 0$

66. $\lim_{n\to\infty} \frac{1}{n+1} = 0$

67. $\lim_{n\to\infty} \frac{1}{n^2} = 0$

68. $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$

69. $\lim_{n\to\infty} \frac{n+1}{n} = 1$

70. $\lim_{n\to\infty} \frac{n}{n+1} = 1$

In Problems 71–82, use the limit laws to determine $\lim_{n\to\infty} a_n = a$.

71. $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n^2}\right)$

72. $\lim_{n\to\infty} \left(\frac{2}{n} - \frac{1}{n^2+1}\right)$

73. $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)$

74. $\lim_{n\to\infty} \left(\frac{2n-3}{n}\right)$

75. $\lim_{n\to\infty} \left(\frac{n^2+1}{n^2}\right)$

76. $\lim_{n\to\infty} \left(\frac{3n^2-5}{n^2}\right)$

77. $\lim_{n\to\infty} \left(\frac{n+1}{n^2-1}\right)$

78. $\lim_{n\to\infty} \left(\frac{n+2}{n^2-4}\right)$

79. $\lim_{n\to\infty} \left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n\right]$

80. $\lim_{n\to\infty} (3^{-n} - 4^{-n})$

81. $\lim_{n\to\infty} \frac{n+2^{-n}}{n}$

82. $\lim_{n\to\infty} \frac{n+3^{-n}}{n}$

■ 2.2.3

In Problems 83–92, the sequence $\{a_n\}$ is recursively defined. Compute a_n for $n = 1, 2, \ldots, 5$.

83. $a_{n+1} = 2a_n, a_0 = 1$

84. $a_{n+1} = 2a_n, a_0 = 3$

85. $a_{n+1} = 3a_n - 2, a_0 = 1$

86. $a_{n+1} = 3a_n - 2, a_0 = 2$

87. $a_{n+1} = 4 - 2a_n, a_0 = 5$

88. $a_{n+1} = 4 - 2a_n, a_0 = \frac{4}{3}$

89. $a_{n+1} = \frac{a_n}{1+a_n}, a_0 = 1$

90. $a_{n+1} = \frac{a_n}{a_n+3}, a_0 = 2$

91. $a_{n+1} = a_n + \frac{1}{a_n}, a_0 = 1$

92. $a_{n+1} = 5a_n - \frac{5}{a_n}, a_0 = 2$

In Problems 93–102, the sequence $\{a_n\}$ is recursively defined. Find all fixed points of $\{a_n\}$.

93. $a_{n+1} = \frac{1}{2}a_n + 2$

94. $a_{n+1} = \frac{1}{3}a_n + \frac{4}{3}$

95. $a_{n+1} = \frac{2}{5}a_n - \frac{9}{5}$

96. $a_{n+1} = -\frac{1}{3}a_n + \frac{1}{4}$

97. $a_{n+1} = \frac{4}{a_n}$

98. $a_{n+1} = \frac{7}{a_n}$

99. $a_{n+1} = \frac{2}{a_n+2}$

100. $a_{n+1} = \frac{3}{a_n-2}$

101. $a_{n+1} = \sqrt{5a_n}$

102. $a_{n+1} = \sqrt{7a_n}$

In Problems 103–110, assume that $\lim_{n\to\infty} a_n$ exists. Find all fixed points of $\{a_n\}$, and use a table or other reasoning to guess which fixed point is the limiting value for the given initial condition.

103. $a_{n+1} = \frac{1}{2}(a_n + 5), a_0 = 1$

104. $a_{n+1} = \frac{1}{3}\left(a_n + \frac{1}{9}\right), a_0 = 1$

105. $a_{n+1} = \sqrt{2a_n}, a_0 = 1$

106. $a_{n+1} = \sqrt{2a_n}, a_0 = 0$

107. $a_{n+1} = 2a_n(1 - a_n), a_0 = 0.1$

108. $a_{n+1} = 2a_n(1 - a_n), a_0 = 0$

109. $a_{n+1} = \frac{1}{2}\left(a_n + \frac{4}{a_n}\right), a_0 = 1$

110. $a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right), a_0 = -1$

■ 2.3 More Population Models

The material presented in this section will be revisited in Section 5.6. Section 2.3 can be postponed until then.

An important biological application of sequences consists of models of seasonally breeding populations with nonoverlapping generations where the population size at one generation depends only on the population size of the previous generation. The exponential growth model of Section 2.1 fits into this category. We denote the population size at time t by $N(t)$ or N_t, $t = 0, 1, 2, \ldots$. To model how the population size at generation $t + 1$ is related to the population size at generation

A rectangle whose sides bear the golden ratio is called a **golden rectangle**; it is thought to be the visually most pleasing proportion a rectangle can have. Golden rectangles were known to the ancient Greeks, who used them to scale the dimensions of their buildings (e.g., the Parthenon). Ratios of successive Fibonacci numbers can be found in nature as well. For instance, the florets on plants such as the sunflower run in spirals, and the ratios of the number of spirals running in opposite directions are often successive Fibonacci numbers.

Section 2.3 Problems

■ 2.3.1

In Problems 1–6, assume that the population growth is described by the Beverton–Holt recruitment curve with growth parameter R and carrying capacity K. For the given values of R and K, graph N_t/N_{t+1} as a function of N_t and find the recursion for the Beverton–Holt recruitment curve.

1. $R = 2, K = 15$
2. $R = 2, K = 50$
3. $R = 1.5, K = 40$
4. $R = 3, K = 120$
5. $R = 2.5, K = 90$
6. $R = 2, K = 150$

In Problems 7–12, assume that the population growth is described by the Beverton–Holt recruitment curve with growth parameter R and carrying capacity K. Find R and K.

7. $N_{t+1} = \dfrac{2N_t}{1 + N_t/20}$
8. $N_{t+1} = \dfrac{3N_t}{1 + 2N_t/40}$
9. $N_{t+1} = \dfrac{1.5N_t}{1 + 0.5N_t/30}$
10. $N_{t+1} = \dfrac{2N_t}{1 + N_t/200}$
11. $N_{t+1} = \dfrac{4N_t}{1 + N_t/150}$
12. $N_{t+1} = \dfrac{5N_t}{1 + N_t/20}$

In Problems 13–18, assume that the population growth is described by the Beverton–Holt recruitment curve with growth parameter R and carrying capacity K. Find all fixed points.

13. $N_{t+1} = \dfrac{4N_t}{1 + N_t/30}$
14. $N_{t+1} = \dfrac{3N_t}{1 + N_t/60}$
15. $N_{t+1} = \dfrac{2N_t}{1 + N_t/30}$
16. $N_{t+1} = \dfrac{2N_t}{1 + N_t/100}$
17. $N_{t+1} = \dfrac{3N_t}{1 + N_t/30}$
18. $N_{t+1} = \dfrac{5N_t}{1 + N_t/120}$

In Problems 19–24, assume that the population growth is described by the Beverton–Holt recruitment curve with growth parameter R and carrying capacity K. Find the population sizes for $t = 1, 2, \ldots, 5$ and find $\lim_{t\to\infty} N_t$ for the given initial value N_0.

19. $R = 2, K = 10, N_0 = 2$
20. $R = 2, K = 20, N_0 = 5$
21. $R = 3, K = 15, N_0 = 1$
22. $R = 3, K = 30, N_0 = 0$
23. $R = 4, K = 40, N_0 = 3$
24. $R = 4, K = 20, N_0 = 10$

■ 2.3.2

In Problems 25–30, assume that the discrete logistic equation is used with parameters R and K. Write the equation in the canonical form $x_{t+1} = rx_t(1 - x_t)$, and determine r and x_t in terms of R, K, and N_t.

25. $R = 1, K = 10$
26. $R = 1, K = 20$
27. $R = 2, K = 15$
28. $R = 2, K = 20$
29. $R = 2.5, K = 30$
30. $R = 2.5, K = 50$

In Problems 31–34, we will investigate the advantage of dimensionless variables.

31. (a) Let N_t denote the population size at time t and let K denote the carrying capacity. Both quantities are measured in units of number of individuals. Show that $x_t = N_t/K$ is dimensionless.

(b) Let M_t denote the population size at time t and let L denote the carrying capacity. Assume that M_t and L are measured in units of 1000 individuals. Show that $y_t = M_t/L$ is dimensionless.

(c) How are N_t and M_t related? How are K and L related?

(d) Use (c) to find M_t and L if there are 20,000 individuals at time t and the carrying capacity is 5000.

(e) Show that, for the population size and the carrying capacity in (d), $x_t = y_t$.

32. To quantify the spatial structure of a plant population, it might be convenient to introduce a characteristic length scale. This length scale might be characterized by the average dispersal distance of the plant under study. Assume that the characteristic length scale is denoted by L. Denote by x the distance of seeds from their source. Define $z = x/L$. Find z if $x = 100$ cm and $L = 50$ cm, and show that z has the same value if x and L are measured in units of meters instead.

33. Suppose a bacterium divides every 20 minutes, which we call the characteristic time scale and denote by T. Let t be the time elapsed since the beginning of an experiment that involves this bacterium. Define $z = t/T$. Find z if $t = 120$ minutes, and show that z has the same value if t and T are measured in units of hours instead.

34. The time to the most recent common ancestor of a pair of individuals from a randomly mating population depends on the population size. Let t denote the time, measured in units of generations, to the most recent common ancestor, and let T be equal to N generations, where N is the population size of the randomly mating population. Define $z = t/T$. Show that z is dimensionless and that the value of z does not change, regardless of whether t and T are measured in units of generations or in units of, say, years. (Assume that one generation is equal to n years.)

In Problems 35–46, we investigate the behavior of the discrete logistic equation

$$x_{t+1} = rx_t(1 - x_t)$$

Compute x_t for $t = 0, 1, 2, \ldots, 20$ for the given values of r and x_0, and graph x_t as a function of t.

35. $r = 2, x_0 = 0.2$
36. $r = 2, x_0 = 0.1$
37. $r = 2, x_0 = 0.9$
38. $r = 2, x_0 = 0$
39. $r = 3.1, x_0 = 0.5$
40. $r = 3.1, x_0 = 0.1$
41. $r = 3.1, x_0 = 0.9$
42. $r = 3.1, x_0 = 0$
43. $r = 3.8, x_0 = 0.5$
44. $r = 3.8, x_0 = 0.1$
45. $r = 3.8, x_0 = 0.9$
46. $r = 3.8, x_0 = 0$

2.3.3

In Problems 47–50, graph the Ricker's curve

$$N_{t+1} = N_t \exp\left[R\left(1 - \frac{N_t}{K}\right)\right]$$

in the N_t–N_{t+1} plane for the given values of R and K. Find the points of intersection of this graph with the line $N_{t+1} = N_t$.

47. $R = 2, K = 10$ **48.** $R = 3, K = 15$

49. $R = 2.5, K = 12$ **50.** $R = 4, K = 20$

In Problems 51–54, we investigate the behavior of the Ricker's curve

$$N_{t+1} = N_t \exp\left[R\left(1 - \frac{N_t}{K}\right)\right]$$

Compute N_t for $t = 1, 2, \ldots, 20$ for the given values of R, K, and N_0, and graph N_t as a function of t.

51. (a) $R = 1, K = 20, N_0 = 5$
(b) $R = 1, K = 20, N_0 = 10$
(c) $R = 1, K = 20, N_0 = 20$ **(d)** $R = 1, K = 20, N_0 = 0$

52. (a) $R = 1.8, K = 20, N_0 = 5$
(b) $R = 1.8, K = 20, N_0 = 10$
(c) $R = 1.8, K = 20, N_0 = 20$ **(d)** $R = 1.8, K = 20, N_0 = 0$

53. (a) $R = 2.1, K = 20, N_0 = 5$
(b) $R = 2.1, K = 20, N_0 = 10$
(c) $R = 2.1, K = 20, N_0 = 20$ **(d)** $R = 2.1, K = 20, N_0 = 0$

54. (a) $R = 2.8, K = 20, N_0 = 5$
(b) $R = 2.8, K = 20, N_0 = 10$
(c) $R = 2.8, K = 20, N_0 = 20$ **(d)** $R = 2.8, K = 20, N_0 = 0$

2.3.4

55. Compute N_t and N_t/N_{t-1} for $t = 2, 3, 4, \ldots, 20$ when

$$N_{t+1} = N_t + N_{t-1}$$

with $N_0 = 1$ and $N_1 = 1$.

56. Compute N_t and N_t/N_{t-1} for $t = 2, 3, 4, \ldots, 20$ when

$$N_{t+1} = N_t + 2N_{t-1}$$

with $N_0 = 1$ and $N_1 = 1$.

57. In the text, an interpretation of the Fibonacci recursion

$$N_{t+1} = N_t + N_{t-1}$$

is given. Use a similar example to give an interpretation of the recursion

$$N_{t+1} = N_t + 2N_{t-1}$$

58. In the text, an interpretation of the Fibonacci recursion

$$N_{t+1} = N_t + N_{t-1}$$

is given. Use a similar example to give an interpretation of the recursion

$$N_{t+1} = 2N_t + N_{t-1}$$

Same

Chapter 2 Key Terms

Discuss the following definitions and concepts:

1. Exponential growth
2. Growth constant
3. Fixed point
4. Equilibrium
5. Recursion
6. Solution
7. Density independence
8. Sequence
9. First-order recursion
10. Limit
11. Long-term behavior
12. Convergence, divergence
13. Limit laws
14. Difference equation
15. Beverton–Holt recruitment curve
16. Density dependence
17. Carrying capacity
18. Growth parameter
19. Discrete logistic equation
20. Nondimensionalization
21. Periodic behavior
22. Chaos
23. Ricker's curve
24. Fibonacci sequence
25. Golden mean

Chapter 2 Review Problems

In Problems 1–10, find the limits.

1. $\lim_{n\to\infty} 2^{-n}$ **2.** $\lim_{n\to\infty} 3^n$

3. $\lim_{n\to\infty} 40(1 - 4^{-n})$ **4.** $\lim_{n\to\infty} \frac{2}{1 + 2^{-n}}$

5. $\lim_{n\to\infty} a^n$ when $a > 1$ **6.** $\lim_{n\to\infty} a^n$ when $0 < a < 1$

7. $\lim_{n\to\infty} \frac{n(n+1)}{n^2 - 1}$ **8.** $\lim_{n\to\infty} \frac{n^2 + n - 6}{n - 2}$

9. $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1}$ **10.** $\lim_{n\to\infty} \frac{n+1}{\sqrt{n}}$

In Problems 11–14, write a_n explicitly as a function of n on the basis of the first five terms of the sequence a_n, $n = 0, 1, 2, \ldots$.

11. $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}$ **12.** $\frac{2}{2}, \frac{6}{4}, \frac{12}{8}, \frac{20}{16}, \frac{30}{32}$

13. $\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}$ **14.** $0, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}$

15. Density-Dependent Growth The Beverton–Holt recruitment curve is given by the recursion

$$N_{t+1} = \frac{RN_t}{1 + \frac{R-1}{K}N_t}$$

where $R > 1$ and $K > 0$. When $N_0 > 0$, $\lim_{t\to\infty} N_t = K$ for all values of $R > 0$. To investigate how R affects the limiting behavior of N_t, find N_t for $t = 1, 2, 3, \ldots, 10$ for $K = 100$ and $N_0 = 20$ when **(a)** $R = 2$, **(b)** $R = 5$, and **(c)** $R = 10$, and plot N_t as a function of t for the three choices of R in one coordinate system.

In Problems 16–18, we discuss population models when the population size at time $t + 1$ depends not only on the population size at time t, but also on the growth conditions at time t, which may vary over time.

16. Temporally Varying Environment The recursion

$$N_{t+1} = R_t N_t$$

describes growth in a temporally varying environment if we interpret R_t as the growth parameter in generation t. A population was followed over 10 years and the population sizes were recorded each year. Use the data provided to find R_t for $t = 0, 1, 2, \ldots, 9$:

t	N_t
0	10
1	15.5
2	15.6
3	10.8
4	15.6
5	32.2
6	95.1
7	103.2
8	165.0
9	418.7
10	15.7

17. Temporally Varying Environment The recursion

$$N_{t+1} = R_t N_t$$

describes growth in a temporally varying environment if we interpret R_t as the growth parameter in generation t. A population was followed over 20 years and the population sizes were recorded every year. The following table provides the population size data and the inferred values of R_t for each of the 20 years:

t	N_t	R_t
0	10.0	2.78
1	27.8	0.29
2	8.10	0.43
3	3.49	0.25
4	0.87	2.90
5	2.52	1.67
6	4.21	1.17
7	4.94	0.69
8	3.39	1.45
9	4.92	1.13
10	5.56	0.08
11	0.45	0.88
12	0.40	2.69
13	1.06	0.36
14	0.38	0.08
15	0.03	2.34
16	0.07	2.13
17	0.15	2.20
18	0.34	2.80
19	0.94	0.29
20	0.28	1.22

The values of N_t indicate that the population heads toward extinction. The long-term behavior of the geometric mean of the growth parameter, denoted by $\hat{R}_t$ (read "R sub t hat"), is defined as

$$\hat{R}_t = (R_0 R_1 \cdots R_{t-1})^{1/t}$$

and determines whether the population will go extinct. Specifically, if

$$\lim_{t\to\infty} \hat{R}_t < 1$$

then the population will go extinct. Compute $\hat{R}_t$ for $t = 1, 2, \ldots, 20$.

18. Temporally Varying Environment The recursion

$$N_{t+1} = R_t N_t$$

describes growth in a temporally varying environment if we interpret R_t as the growth parameter in generation t.

(a) Show that

$$N_t = (R_{t-1} R_{t-2} \cdots R_1 R_0) N_0$$

(b) The quantity $\hat{R}_t$ (read "R sub t hat"), defined as

$$\hat{R}_t = (R_{t-1} R_{t-2} \cdots R_1 R_0)^{1/t}$$

is called the **geometric mean.** Show that

$$N_t = (\hat{R}_t)^t N_0$$

(c) The **arithmetic mean** of a sequence of numbers $x_0, x_1, \ldots, x_{n-1}$ is defined as

$$\bar{x}_n = \frac{x_0 + x_1 + \cdots + x_{n-1}}{n}$$

Set $r_t = \ln R_t$ and show that

$$\bar{r}_t = \frac{\ln R_{t-1} + \ln R_{t-2} + \cdots + \ln R_0}{t}$$

(d) Use (c) to show that

$$N_t = N_0 e^{\bar{r}_t t}$$

19. Harvesting Model Let N_t denote the population size at time t, and assume that

$$N_{t+1} = (1-c)N_t \exp\left[R\left(1 - \frac{(1-c)N_t}{K}\right)\right]$$

where R and K are positive constants and c is the fraction harvested. Find N_t for $t = 1, 2, \ldots, 20$ when $R = 1$, $K = 100$, and $N_0 = 50$ for **(a)** $c = 0.1$, **(b)** $c = 0.5$, and **(c)** $c = 0.9$.

20. Harvesting Model Let N_t denote the population size at time t, and assume that

$$N_{t+1} = (1-c)N_t \exp\left[R\left(1 - \frac{(1-c)N_t}{K}\right)\right]$$

where R and K are positive constants and c is the fraction harvested. Find N_t for $t = 1, 2, \ldots, 20$ when $R = 3$, $K = 100$, and $N_0 = 50$ for **(a)** $c = 0.1$, **(b)** $c = 0.5$, and **(c)** $c = 0.9$.

done