offspring, regardless of the current population density. Such growth is called density independent.

When \( R > 1 \), it follows that \( 1/R \), the parent–offspring ratio, is less than 1, implying that the number of offspring exceeds the number of parents. Density-independent growth with \( R > 1 \) results in an ever-increasing population size. This model eventually becomes biologically unrealistic, since any population will sooner or later experience food or habitat limitations that will limit its growth. (We will discuss models that include such limitations in Section 2.3.)

The density independence in exponential growth is reflected in a graph of \( N_t/N_{t+1} \) as a function of \( N_t \), which is a horizontal line at level \( 1/R \) (Figure 2.7).

As before, only a selected number of points are realized on the graph of \( N_t/N_{t+1} \) as a function of \( N_t \), and time is implicit in the graph. (See Figure 2.8, with \( R = 2 \) and \( N_0 = 1 \).)

![Figure 2.7](image_url)  
Figure 2.7 The graph of the parent–offspring ratio \( N_t/N_{t+1} \) as a function of \( N_t \) when \( N_t > 0 \).

![Figure 2.8](image_url)  
Figure 2.8 The graph of the parent–offspring ratio \( N_t/N_{t+1} \) as a function of \( N_t \), when \( N_t = 1 \) and \( R = 2 \).

### Section 2.1 Problems

In Problems 1–4, produce a table for \( t = 0, 1, 2, \ldots, 5 \) and graph the function \( N_t \).

1. \( N_t = 3^t \)
2. \( N_t = 10 \cdot 2^t \)
3. \( N_t = \frac{25}{4^t} \)
4. \( N_t = (0.3)(0.9)^t \)

In Problems 5–10, give a formula for \( N(t) \), \( t = 0, 1, 2, \ldots \), on the basis of the information provided.

5. \( N_0 = 2 \); population doubles every 20 minutes; one unit of time is 20 minutes
6. \( N_0 = 4 \); population doubles every 40 minutes; one unit of time is 40 minutes
7. \( N_0 = 1 \); population doubles every 40 minutes; one unit of time is 80 minutes
8. \( N_0 = 6 \); population doubles every 40 minutes; one unit of time is 60 minutes
9. \( N_0 = 2 \); population quadruples every 30 minutes; one unit of time is 15 minutes
10. \( N_0 = 10 \); population quadruples every 20 minutes; one unit of time is 10 minutes
11. Suppose \( N_t = 20 \cdot 4^t \), \( t = 0, 1, 2, \ldots \), and one unit of time corresponds to 3 hours. Determine the amount of time it takes the population to double in size.
12. Suppose \( N_t = 100 \cdot 2^t \), \( t = 0, 1, 2, \ldots \), and one unit of time corresponds to 2 hours. Determine the amount of time it takes the population to triple in size.

13. A strain of bacteria reproduces asexually every hour. That is, every hour, each bacterial cell splits into two cells. If, initially, there is one bacterium, find the number of bacterial cells after 1 hour, 2 hours, 3 hours, 4 hours, and 5 hours.

14. A strain of bacteria reproduces asexually every 30 minutes. That is, every 30 minutes, each bacterial cell splits into two cells. If, initially, there is one bacterium, find the number of bacterial cells after 1 hour, 2 hours, 3 hours, 4 hours, and 5 hours.

15. A strain of bacteria reproduces asexually every 23 minutes. That is, every 23 minutes, each bacterial cell splits into two cells. If, initially, there is 1 bacterium, how long will it take until there are 128 bacteria?

16. A strain of bacteria reproduces asexually every 42 minutes. That is, every 42 minutes, each bacterial cell splits into two cells. If, initially, there is 1 bacterium, how long will it take until there are 512 bacteria?

17. A strain of bacteria reproduces asexually every 10 minutes. That is, every 10 minutes, each bacterial cell splits into two cells. If, initially, there are 3 bacteria, how long will it take until there are 96 bacteria?

18. A strain of bacteria reproduces asexually every 50 minutes. That is, every 50 minutes, each bacterial cell splits into two cells. If, initially, there are 10 bacteria, how long will it take until there are 640 bacteria?
19. Find the exponential growth equation for a population that doubles in size every unit of time and that has 40 individuals at time 0.
20. Find the exponential growth equation for a population that doubles in size every unit of time and that has 53 individuals at time 0.
21. Find the exponential growth equation for a population that triples in size every unit of time and that has 20 individuals at time 0.
22. Find the exponential growth equation for a population that triples in size every unit of time and that has 72 individuals at time 0.
23. Find the exponential growth equation for a population that quadruples in size every unit of time and that has 5 individuals at time 0.
24. Find the exponential growth equation for a population that quadruples in size every unit of time and that has 17 individuals at time 0.
25. Find the recursion for a population that doubles in size every unit of time and that has 20 individuals at time 0.
26. Find the recursion for a population that doubles in size every unit of time and that has 37 individuals at time 0.
27. Find the recursion for a population that triples in size every unit of time and that has 10 individuals at time 0.
28. Find the recursion for a population that triples in size every unit of time and that has 84 individuals at time 0.
29. Find the recursion for a population that quadruples in size every unit of time and that has 30 individuals at time 0.
30. Find the recursion for a population that quadruples in size every unit of time and that has 62 individuals at time 0.

In Problems 31-34, graph the functions \( f(x) = a^x \), \( x \in [0, \infty) \), and \( N_t = R^t \), \( t \in \mathbb{N} \), together in one coordinate system for the indicated values of \( a \) and \( R \).
31. \( a = R = 2 \)  
32. \( a = R = 3 \)  
33. \( a = R = 1/2 \)  
34. \( a = R = 1/3 \)  

In Problems 35-46, find the population sizes for \( t = 0, 1, 2, \ldots, 5 \) for each recursion.
35. \( N_{t+1} = 2N_t \), with \( N_0 = 3 \)  
36. \( N_{t+1} = 2N_t \), with \( N_0 = 5 \)  
37. \( N_{t+1} = 3N_t \), with \( N_0 = 2 \)  
38. \( N_{t+1} = 3N_t \), with \( N_0 = 7 \)  
39. \( N_{t+1} = 5N_t \), with \( N_0 = 1 \)  
40. \( N_{t+1} = 7N_t \), with \( N_0 = 4 \)  
41. \( N_{t+1} = \frac{1}{2}N_t \), with \( N_0 = 1024 \)  
42. \( N_{t+1} = \frac{1}{2}N_t \), with \( N_0 = 4096 \)  
43. \( N_{t+1} = \frac{1}{3}N_t \), with \( N_0 = 729 \)  
44. \( N_{t+1} = \frac{1}{3}N_t \), with \( N_0 = 3645 \)  
45. \( N_{t+1} = \frac{1}{3}N_t \), with \( N_0 = 31250 \)  
46. \( N_{t+1} = \frac{1}{4}N_t \), with \( N_0 = 8192 \)  

In Problems 47-58, write \( N_t \) as a function of \( t \) for each recursion.
47. \( N_{t+1} = 2N_t \), with \( N_0 = 15 \)  
48. \( N_{t+1} = 2N_t \), with \( N_0 = 7 \)  
49. \( N_{t+1} = 3N_t \), with \( N_0 = 12 \)  
50. \( N_{t+1} = 3N_t \), with \( N_0 = 3 \)  
51. \( N_{t+1} = 4N_t \), with \( N_0 = 24 \)  
52. \( N_{t+1} = 5N_t \), with \( N_0 = 17 \)  
53. \( N_{t+1} = \frac{1}{2}N_t \), with \( N_0 = 5000 \)  
54. \( N_{t+1} = \frac{1}{2}N_t \), with \( N_0 = 2300 \)  
55. \( N_{t+1} = \frac{1}{2}N_t \), with \( N_0 = 8000 \)  
56. \( N_{t+1} = \frac{1}{2}N_t \), with \( N_0 = 3500 \)  
57. \( N_{t+1} = \frac{1}{2}N_t \), with \( N_0 = 1200 \)  
58. \( N_{t+1} = \frac{1}{2}N_t \), with \( N_0 = 6400 \)

In Problems 59-66, graph the line \( N_{t+1} = RN_t \) in the \( N_t - N_{t+1} \) plane for the indicated value of \( R \) and locate the points \( (N_t, N_{t+1}) \), \( t = 0, 1, 2 \) for the given value of \( N_0 \).
59. \( R = 2, N_0 = 2 \)  
60. \( R = 2, N_0 = 3 \)  
61. \( R = 3, N_0 = 1 \)  
62. \( R = 4, N_0 = 2 \)  
63. \( R = \frac{1}{2}, N_0 = 16 \)  
64. \( R = \frac{1}{2}, N_0 = 64 \)  
65. \( R = \frac{3}{4}, N_0 = 81 \)  
66. \( R = \frac{3}{4}, N_0 = 16 \)

In Problems 67-74, graph the line \( N_{t+1} = \frac{1}{R}N_t \) in the \( N_t - N_{t+1} \) plane for the indicated value of \( R \) and locate the points \( (N_t, N_{t+1}) \), \( t = 0, 1, 2 \) for the given value of \( N_0 \). Find the parent-offspring ratio.
67. \( R = 2, N_0 = 2 \)  
68. \( R = 2, N_0 = 4 \)  
69. \( R = 3, N_0 = 2 \)  
70. \( R = 4, N_0 = 1 \)  
71. \( R = \frac{1}{2}, N_0 = 16 \)  
72. \( R = \frac{1}{2}, N_0 = 128 \)  
73. \( R = \frac{3}{4}, N_0 = 27 \)  
74. \( R = \frac{3}{4}, N_0 = 64 \)

75. A bird population lives in a habitat where the number of nesting sites is a limiting factor in population growth. In which of the following cases would you expect that the growth of this bird population over the next few generations could be reasonably well approximated by exponential growth?
(a) All nesting sites are occupied.
(b) The bird population just invaded the habitat, and the population size is still much smaller than the available nesting sites.
(c) In the previous year, a hurricane killed more than 90% of the birds in this habitat.

76. Pollen records show that the number of Scotch pine \( (P. sylvestris) \) grew exponentially for about 500 years after colonization of the Norfolk region of Great Britain about 9500 years ago. Can you find a possible explanation for this growth?
77. Exponential growth generally occurs when population growth is density independent. List conditions under which a population might stop growing exponentially.

2.2 Sequences

2.2.1 What Are Sequences?

Before we explore other discrete-time population models, we need to develop further the theory of functions with domain \( \mathbb{N} \). The functions are of the form

\[
\begin{align*}
  f : \mathbb{N} &\rightarrow \mathbb{R} \\
  n &\rightarrow f(n)
\end{align*}
\]
Thus, $a_3$ is the same as $a_1$, $a_4$ is the same as $a_2$ and hence $a_0$, and so on.

The last two examples illustrate that fixed points are only candidates for limits and that, depending on the initial condition, the sequence $\{a_n\}$ may or may not converge to a given fixed point. If we know, however, that a sequence $\{a_n\}$ does converge, then the limit of the sequence must be one of the fixed points.

There is a graphical method for finding fixed points, which we will mention briefly here: If the recursion is of the form $a_{n+1} = g(a_n)$, then a fixed point satisfies $a = g(a)$. This suggests that if we graph $y = g(x)$ and $y = x$ in the same coordinate system, then fixed points are located where the two graphs intersect, as shown in Figure 2.13.

![Figure 2.13 A graphical way to find fixed points. (See text for explanation.)](image)

We will return to the relationship between fixed points and limits in Sectic 5.6, where we will learn methods that allow us to determine whether a sequence converges to a particular fixed point.

**Section 2.2 Problems**

**2.2.1**

In Problems 1–16, determine the values of the sequence $\{a_n\}$ for $n = 0, 1, 2, \ldots, 5$.

1. $a_n = n$
2. $a_n = 3n^2$
3. $a_n = \frac{n}{n + 2}$
4. $f(n) = \frac{1}{n + 2}$
5. $f(n) = \frac{1}{n + 1}$
6. $f(n) = (n + 1)^2$
7. $a_n = \frac{(-1)^n}{n + 1}$
8. $a_n = \frac{n^2}{n + 1}$
9. $a_n = \frac{1}{n + 1}$
10. $a_n = \frac{n}{n + 1}$
11. $a_n = \frac{1}{3}$
12. $a_n = \frac{1}{2^n}$

In Problems 17–24, find the next four values of the sequence $\{a_n\}$ on the basis of the values of $a_0, a_1, a_2, \ldots, a_5$.

17. $1, 2, 3, 4, 5$
18. $0, 1, 2, 3, 4, 5$
19. $1, 2, 3, 4, 5$
20. $-1, 1, 2, 3, 4, 5$
21. $1, 2, 3, 4, 5$
22. $1, 2, 3, 4, 5$
23. $\sqrt{1 + e}, \sqrt{2 + e^2}, \sqrt{3 + e^3}, \sqrt{4 + e^4}, \sqrt{5 + e^5}$
24. $\sin \frac{\pi}{2}, -\sin \frac{\pi}{4}, \sin \frac{\pi}{6}, -\sin \frac{\pi}{8}, \sin \frac{\pi}{10}$

In Problems 25–36, find an expression for $a_n$ on the basis of $a_0, a_1, a_2, \ldots$.

25. $0, 0, 0, 0, \ldots$
26. $0, 0, 0, 0, \ldots$
27. $1, 2, 4, 8, 16, \ldots$
28. $1, 3, 5, 7, 9, \ldots$
29. $1, 2, 4, 8, \ldots$
30. $1, 2, 4, 8, 16, \ldots$
31. $-1, 2, -3, 4, -5, \ldots$
32. $-1, 2, -3, 4, -5, \ldots$
33. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
34. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$
35. $\sin(\pi), \sin(2\pi), \sin(3\pi), \sin(4\pi), \sin(5\pi), \ldots$
36. $-\cos \frac{\pi}{2}, -\cos \frac{\pi}{4}, -\cos \frac{\pi}{6}, -\cos \frac{\pi}{8}, -\cos \frac{\pi}{10}, \ldots$

**2.2.2**

In Problems 37–44, write the first five terms of the sequence $\{a_n\}$, $n = 0, 1, 2, \ldots$, and find $\lim_{n \to \infty} a_n$.

37. $a_n = \frac{1}{n + 1}$
38. $a_n = \frac{2}{n + 1}$
39. $a_n = \frac{n}{n + 1}$
40. $a_n = \frac{2n}{n + 2}$
41. $a_n = \frac{1}{n^2 + 1}$
42. $a_n = \frac{1}{\sqrt{n} + 1}$
43. \( a_n = \frac{(-1)^n}{n+1} \)
44. \( a_n = \frac{(-1)^n}{n^3+3} \)

In Problems 43–52, write the first five terms of the sequence \( \{a_n\} \), \( n = 0, 1, 2, 3, \ldots \), and determine whether \( \lim_{n \to \infty} a_n \) exists. If the limit exists, find it.

45. \( a_n = \frac{n^2}{n+1} \)
46. \( a_n = \frac{n^3}{n+1} \)
47. \( a_n = \sqrt{n} \)
48. \( a_n = n^2 \)
49. \( a_n = 2^n \)
50. \( a_n = \left(\frac{1}{2}\right)^n \)
51. \( a_n = 3^n \)
52. \( a_n = \left(\frac{1}{3}\right)^n \)

**Formal Definition of Limits:**
In Problems 53–64, \( \lim_{n \to \infty} a_n = a \).
Find the limit \( a \) and determine \( N \) so that \( |a_n - a| < \epsilon \) for all \( n > N \) for the given value of \( \epsilon \).

53. \( a_n = \frac{1}{n}, \epsilon = 0.01 \)
54. \( a_n = \frac{1}{n}, \epsilon = 0.02 \)
55. \( a_n = \frac{1}{n}, \epsilon = 0.001 \)
56. \( a_n = \frac{1}{n}, \epsilon = 0.0001 \)
57. \( a_n = \frac{1}{\sqrt{n}}, \epsilon = 0.1 \)
58. \( a_n = \frac{1}{\sqrt{n}}, \epsilon = 0.05 \)
59. \( a_n = \frac{(-1)^n}{n}, \epsilon = 0.01 \)
60. \( a_n = \frac{(-1)^n}{n}, \epsilon = 0.001 \)
61. \( a_n = \frac{n+1}{n}, \epsilon = 0.01 \)
62. \( a_n = \frac{n+1}{n}, \epsilon = 0.05 \)
63. \( a_n = \frac{n^2}{n^2+1}, \epsilon = 0.01 \)
64. \( a_n = \frac{n^2}{n^2+1}, \epsilon = 0.001 \)

**Formal Definition of Limits:**
In Problems 65–70, use the formal definition of limits to show that \( \lim_{n \to \infty} a_n = a \); that is, find \( N \) such that for every \( \epsilon > 0 \), there exists an \( N \) such that \( |a_n - a| < \epsilon \) whenever \( n > N \).

65. \( \lim_{n \to \infty} \frac{n}{n+1} = 1 \)
66. \( \lim_{n \to \infty} \frac{1}{n+1} = 0 \)
67. \( \lim_{n \to \infty} \frac{n}{n^2} = 0 \)
68. \( \lim_{n \to \infty} \frac{1}{n^2+1} = 0 \)
69. \( \lim_{n \to \infty} \frac{n+1}{n} = 1 \)
70. \( \lim_{n \to \infty} \frac{n}{n+1} = 1 \)

In Problems 71–82, use the limit laws to determine \( \lim_{n \to \infty} a_n = a \).

71. \( \lim_{n \to \infty} \left(\frac{n+1}{n^2}\right) \)
72. \( \lim_{n \to \infty} \left(\frac{n^2+1}{n^2}\right) \)
73. \( \lim_{n \to \infty} \left(\frac{n+1}{n}\right) \)
74. \( \lim_{n \to \infty} \left(\frac{3n+2}{n+3}\right) \)
75. \( \lim_{n \to \infty} \left(\frac{n^2+1}{n^2}\right) \)
76. \( \lim_{n \to \infty} \left(\frac{3n^2+3}{n^2}\right) \)
77. \( \lim_{n \to \infty} \left(\frac{n+1}{n^3+3}\right) \)
78. \( \lim_{n \to \infty} \left(\frac{n+2}{n^2-4}\right) \)
79. \( \lim_{n \to \infty} \left(\frac{\left(\frac{1}{3}\right)^n}{\left(\frac{1}{2}\right)^n}\right) \)
80. \( \lim_{n \to \infty} \left(\frac{n+2}{n^3-4}\right) \)
81. \( \lim_{n \to \infty} \left(\frac{n+2}{n+3}\right) \)
82. \( \lim_{n \to \infty} \left(\frac{n^3+1}{n}\right) \)

**2.2.3 More Population Models**

The material presented in this section will be revisited in Section 5.6. Section 2.3 can be postponed until then.

An important biological application of sequences consists of models of seasonally breeding populations with nonoverlapping generations where the population size at one generation depends only on the population size of the previous generation. The exponential growth model of Section 2.1 fits into this category. We denote the population size at time \( t \) by \( N(t) \) or \( N_t \), \( t = 0, 1, 2, \ldots \). To model how the population size at generation \( t + 1 \) is related to the population size at generation
Section 2.3 Problems

2.3.1

In Problems 1–6, assume that the population growth is described by the Beverton–Holt recruitment curve with growth parameter R and carrying capacity K. For the given values of R and K, graph \( N_t / N_{t+1} \) as a function of \( N_t \) and find the recursion for the Beverton–Holt recruitment curve.

1. \( R = 2, \ K = 15 \)  
2. \( R = 2, \ K = 50 \)  
3. \( R = 1.5, \ K = 40 \)  
4. \( R = 3, \ K = 120 \)  
5. \( R = 2.5, \ K = 90 \)  
6. \( R = 2, \ K = 150 \)

In Problems 7–12, assume that the population growth is described by the Beverton–Holt recruitment curve with growth parameter R and carrying capacity K. Find R and K.

7. \( N_{t+1} = \frac{2N_t}{1 + N_t/20} \)  
8. \( N_{t+1} = \frac{3N_t}{1 + 2N_t/40} \)  
9. \( N_{t+1} = \frac{1.5N_t}{1 + 0.5N_t/30} \)  
10. \( N_{t+1} = \frac{2N_t}{1 + N_t/200} \)  
11. \( N_{t+1} = \frac{4N_t}{1 + N_t/150} \)  
12. \( N_{t+1} = \frac{5N_t}{1 + N_t/20} \)

In Problems 13–18, assume that the population growth is described by the Beverton–Holt recruitment curve with growth parameter R and carrying capacity K. Find all fixed points.

13. \( N_{t+1} = \frac{4N_t}{1 + N_t/30} \)  
14. \( N_{t+1} = \frac{3N_t}{1 + N_t/60} \)  
15. \( N_{t+1} = \frac{2N_t}{1 + N_t/30} \)  
16. \( N_{t+1} = \frac{2N_t}{1 + N_t/100} \)  
17. \( N_{t+1} = \frac{3N_t}{1 + N_t/30} \)  
18. \( N_{t+1} = \frac{5N_t}{1 + N_t/120} \)

In Problems 19–24, assume that the population growth is described by the Beverton–Holt recruitment curve with growth parameter R and carrying capacity K. Find the population sizes for \( t = 1, 2, \ldots, S \) and find \( \lim_{t \to \infty} N_t \) for the given initial value \( N_0 \).

19. \( R = 2, \ K = 10, \ N_0 = 2 \)  
20. \( R = 2, \ K = 20, \ N_0 = 5 \)  
21. \( R = 3, \ K = 15, \ N_0 = 1 \)  
22. \( R = 3, \ K = 30, \ N_0 = 0 \)  
23. \( R = 4, \ K = 40, \ N_0 = 3 \)  
24. \( R = 4, \ K = 20, \ N_0 = 10 \)

2.3.2

In Problems 25–30, assume that the discrete logistic equation is used with parameters R and K. Write the equation in the canonical form \( x_{t+1} = rx_t(1 - x_t) \), and determine R and x, in terms of R, K, and \( N_t \).

25. \( R = 1, \ K = 10 \)  
26. \( R = 1, \ K = 20 \)  
27. \( R = 2, \ K = 15 \)  
28. \( R = 2, \ K = 20 \)  
29. \( R = 2.5, \ K = 30 \)  
30. \( R = 2.5, \ K = 50 \)

In Problems 31–34, we will investigate the advantage of dimensionless variables.

31. (a) Let \( N_t \) denote the population size at time \( t \) and let \( K \) denote the carrying capacity. Both quantities are measured in units of number of individuals. Show that \( x_t = N_t / K \) is dimensionless.

(b) Let \( M_t \) denote the population size at time \( t \) and let \( L \) denote the carrying capacity. Assume that \( M_t \) and \( L \) are measured in units of 1000 individuals. Show that \( y_t = M_t / L \) is dimensionless.

(c) How are \( N_t \) and \( M_t \) related? How are \( K \) and \( L \) related?

(d) Use (c) to find \( M_t \) and \( L \) if there are 20,000 individuals at time \( t \) and the carrying capacity is 5000.

(e) Show that, for the population size and the carrying capacity in (d), \( x_t = y_t \).

32. To quantify the spatial structure of a plant population, it might be convenient to introduce a characteristic length scale. This length scale might be characterized by the average dispersal distance of the plant under study. Assume that the characteristic length scale is denoted by \( L \). Denote by \( x \) the distance of seeds from their source. Define \( z = x / L \). Find \( z \) if \( x = 100 \) cm and \( L = 50 \) cm, and show that \( z \) has the same value if \( x \) and \( L \) are measured in units of meters instead.

33. Suppose a bacterium divides every 20 minutes, which we call the characteristic time scale and denote by \( T \). Let \( t \) be the time elapsed since the beginning of an experiment that involves this bacterium. Define \( z = t / T \). Find \( z \) if \( t = 120 \) minutes, and show that \( z \) has the same value if \( t \) and \( T \) are measured in units of hours instead.

34. The time to the most recent common ancestor of a pair of individuals from a randomly mating population depends on the population size. Let \( t \) denote the time, measured in units of generations, to the most recent common ancestor, and let \( T \) be equal to \( N \) generations, where \( N \) is the population size of the randomly mating population. Define \( z = t / T \). Show that \( z \) is dimensionless and that the value of \( z \) does not change, regardless of whether \( t \) and \( T \) are measured in units of generations or in units of, say, years. (Assume that one generation is equal to \( n \) years.)

In Problems 35–46, we investigate the behavior of the discrete logistic equation

\[
x_{t+1} = rx_t(1 - x_t)
\]

Compute \( x_t \) for \( t = 0, 1, 2, \ldots, 20 \) for the given values of \( r \) and \( x_0 \) and graph \( x_t \) as a function of \( t \).

35. \( r = 2, x_0 = 0.2 \)
36. \( r = 2, x_0 = 0.1 \)
37. \( r = 2, x_0 = 0.9 \)
38. \( r = 2, x_0 = 0 \)
39. \( r = 3.1, x_0 = 0.5 \)
40. \( r = 3.1, x_0 = 0.1 \)
41. \( r = 3.1, x_0 = 0.9 \)
42. \( r = 3.1, x_0 = 0 \)
43. \( r = 3.8, x_0 = 0.5 \)
44. \( r = 3.8, x_0 = 0.1 \)
45. \( r = 3.8, x_0 = 0.9 \)
46. \( r = 3.8, x_0 = 0 \)
2.3.3
In Problems 47–50, graph the Ricker’s curve

\[ N_{t+1} = N_t \exp \left[ R \left( 1 - \frac{N_t}{K} \right) \right] \]

in the \( N_t - N_{t+1} \) plane for the given values of \( R \) and \( K \). Find the points of intersection of this graph with the line \( N_{t+1} = N_t \).

47. \( R = 2, K = 10 \)  
48. \( R = 3, K = 15 \)  
49. \( R = 2.5, K = 12 \)  
50. \( R = 4, K = 20 \)

In Problems 51–54, we investigate the behavior of the Ricker’s curve

\[ N_{t+1} = N_t \exp \left[ R \left( 1 - \frac{N_t}{K} \right) \right] \]

Compute \( N_t \) for \( t = 1, 2, \ldots, 20 \) for the given values of \( R \), \( K \), and \( N_0 \), and graph \( N_t \) as a function of \( t \).

51. (a) \( R = 1, K = 20, N_0 = 5 \)  
(b) \( R = 1, K = 20, N_0 = 10 \)  
(c) \( R = 1, K = 20, N_0 = 20 \)  
(d) \( R = 1, K = 20, N_0 = 50 \)

52. (a) \( R = 1.8, K = 20, N_0 = 5 \)  
(b) \( R = 1.8, K = 20, N_0 = 10 \)  
(c) \( R = 1.8, K = 20, N_0 = 20 \)  
(d) \( R = 1.8, K = 20, N_0 = 50 \)

53. (a) \( R = 2.1, K = 20, N_0 = 5 \)  
(b) \( R = 2.1, K = 20, N_0 = 10 \)  
(c) \( R = 2.1, K = 20, N_0 = 20 \)  
(d) \( R = 2.1, K = 20, N_0 = 50 \)

54. (a) \( R = 2.8, K = 20, N_0 = 5 \)  
(b) \( R = 2.8, K = 20, N_0 = 10 \)  
(c) \( R = 2.8, K = 20, N_0 = 20 \)  
(d) \( R = 2.8, K = 20, N_0 = 0 \)

2.3.4
55. Compute \( N_t \) and \( N_t/N_{t-1} \) for \( t = 2, 3, 4, \ldots, 20 \) when

\[ N_{t+1} = N_t + N_{t-1} \]

with \( N_0 = 1 \) and \( N_1 = 1 \).

56. Compute \( N_t \) and \( N_t/N_{t-1} \) for \( t = 2, 3, 4, \ldots, 20 \) when

\[ N_{t+1} = N_t + 2N_{t-1} \]

with \( N_0 = 1 \) and \( N_1 = 1 \).

57. In the text, an interpretation of the Fibonacci recursion

\[ N_{t+1} = N_t + N_{t-1} \]

is given. Use a similar example to give an interpretation of the recursion

\[ N_{t+1} = N_t + 2N_{t-1} \]

58. In the text, an interpretation of the Fibonacci recursion

\[ N_{t+1} = N_t + N_{t-1} \]

is given. Use a similar example to give an interpretation of the recursion

\[ N_{t+1} = 2N_t + N_{t-1} \]

Chapter 2 Key Terms

Discuss the following definitions and concepts:

1. Exponential growth
2. Growth constant
3. Fixed point
4. Equilibrium
5. Recursion
6. Solution
7. Density independence
8. Sequence
9. First-order recursion
10. Limit
11. Long-term behavior
12. Convergence, divergence
13. Limit laws
14. Difference equation
15. Beverton–Holt recruitment curve
16. Density dependence
17. Carrying capacity
18. Growth parameter
19. Discrete logistic equation
20. Nondimensionalization
21. Periodic behavior
22. Chaos
23. Ricker’s curve
24. Fibonacci sequence
25. Golden mean

Chapter 2 Review Problems

In Problems 1–10, find the limits.

1. \( \lim_{n \to \infty} 2^n \)  
2. \( \lim_{n \to \infty} 3^n \)  
3. \( \lim_{n \to \infty} 4(1 - 4^{-n}) \)  
4. \( \lim_{n \to \infty} \frac{2}{1 + 2^{-n}} \)  
5. \( \lim_{n \to \infty} a^n \) when \( a > 1 \)  
6. \( \lim_{n \to \infty} a^n \) when \( 0 < a < 1 \)  
7. \( \lim_{n \to \infty} \frac{n(n+1)}{n^2 - 1} \)  
8. \( \lim_{n \to \infty} \frac{n^2 + n - 6}{n - 2} \)  
9. \( \lim_{n \to \infty} \frac{\sqrt{n}}{n + 1} \)  
10. \( \lim_{n \to \infty} \frac{n + 1}{\sqrt{n}} \)

In Problems 11–14, write \( a_n \) explicitly as a function of \( n \) on the basis of the first five terms of the sequence \( a_n \), \( n = 0, 1, 2, \ldots \).

11. \( 1 \quad 3 \quad 5 \quad 7 \quad 9 \)   
12. \( 2 \quad 6 \quad 12 \quad 20 \quad 30 \)   
13. \( 1 \quad 2 \quad 3 \quad 4 \quad 5 \)   
14. \( 2 \quad 5 \quad 9 \quad 13 \quad 16 \)   

15. Density-Dependent Growth The Beverton–Holt recruitment curve is given by the recursion

\[ N_{t+1} = \frac{RN_t}{1 + \frac{R}{K} N_t} \]

where \( R > 1 \) and \( K > 0 \). When \( N_0 > 0, \lim_{n \to \infty} N_t = K \) for all values of \( R > 0 \). To investigate how \( R \) affects the limiting behavior of \( N_t \), find \( N_t \) for \( t = 1, 2, 3, \ldots, 10 \) for \( K = 100 \) and \( N_0 = 20 \) when (a) \( R = 2 \), (b) \( R = 5 \), and (c) \( R = 10 \), and plot \( N_t \) as a function of \( t \) for the three choices of \( R \) in one coordinate system.

In Problems 16–18, we discuss population models when the population size at time \( t + 1 \) depends not only on the population size at time \( t \), but also on the growth conditions at time \( t \), which may vary over time.

16. Temporally Varying Environment The recursion

\[ N_{t+1} = R/N_t \]
describes growth in a temporally varying environment if we interpret \( R_t \) as the growth parameter in generation \( t \). A population was followed over 10 years and the population sizes were recorded each year. Use the data provided to find \( R_t \) for \( t = 0, 1, 2, \ldots, 9 \):\[
\begin{array}{ccc}
 t & N_t \\
 0 & 10 \\
 1 & 15.5 \\
 2 & 15.6 \\
 3 & 10.8 \\
 4 & 15.6 \\
 5 & 32.2 \\
 6 & 95.1 \\
 7 & 103.2 \\
 8 & 165.0 \\
 9 & 418.7 \\
 10 & 15.7 \\
\end{array}
\]

17. Temporally Varying Environment The recursion
\[
N_{t+1} = R_t N_t
\]
describes growth in a temporally varying environment if we interpret \( R_t \) as the growth parameter in generation \( t \). A population was followed over 20 years and the population sizes were recorded every year. The following table provides the population size data and the inferred values of \( R_t \) for each of the 20 years:

\[
\begin{array}{ccc}
 t & N_t & R_t \\
 0 & 10.0 & 2.78 \\
 1 & 27.8 & 0.29 \\
 2 & 8.10 & 0.43 \\
 3 & 3.49 & 0.25 \\
 4 & 0.87 & 2.90 \\
 5 & 2.52 & 1.67 \\
 6 & 4.21 & 1.17 \\
 7 & 4.94 & 0.69 \\
 8 & 3.39 & 1.45 \\
 9 & 4.92 & 1.13 \\
 10 & 5.56 & 0.08 \\
 11 & 0.45 & 0.88 \\
 12 & 0.40 & 2.69 \\
 13 & 1.06 & 0.36 \\
 14 & 0.38 & 0.08 \\
 15 & 0.03 & 2.34 \\
 16 & 0.07 & 2.13 \\
 17 & 0.15 & 2.20 \\
 18 & 0.34 & 2.80 \\
 19 & 0.94 & 0.29 \\
 20 & 0.28 & 1.22 \\
\end{array}
\]

The values of \( N_t \) indicate that the population heads toward extinction. The long-term behavior of the geometric mean of the growth parameter, denoted by \( \hat{R} \) (read "R sub t hat"), is defined as
\[
\hat{R} = \left( R_0 R_1 \cdots R_{t-1} \right)^{1/t}
\]
and determines whether the population will go extinct. Specifically, if
\[
\lim_{t \to \infty} \hat{R} < 1
\]
then the population will go extinct. Compute \( \hat{R} \) for \( t = 1, 2, \ldots, 20 \).

18. Temporally Varying Environment The recursion
\[
N_{t+1} = R_t N_t
\]
describes growth in a temporally varying environment if we interpret \( R_t \) as the growth parameter in generation \( t \).

(a) Show that
\[
N_t = (R_{t-1} R_{t-2} \cdots R R_0) N_0
\]
(b) The quantity \( \hat{R} \), (read "R sub t hat"), defined as
\[
\hat{R} = (R_{t-1} R_{t-2} \cdots R R_0)^{1/t}
\]
is called the geometric mean. Show that
\[
N_t = (\hat{R})^t N_0
\]
(c) The arithmetic mean of a sequence of numbers \( x_0, x_1, \ldots, x_{n-1} \) is defined as
\[
\bar{x} = \frac{x_0 + x_1 + \cdots + x_{n-1}}{n}
\]
Set \( r_t = \ln R_t \) and show that
\[
\bar{r} = \frac{\ln R_{t-1} + \ln R_{t-2} + \cdots + \ln R_0}{t}
\]
(d) Use (c) to show that
\[
N_t = N_0 e^{\bar{r} t}
\]

19. Harvesting Model Let \( N_t \) denote the population size at time \( t \), and assume that
\[
N_{t+1} = (1 - c) N_t \exp \left[ R \left( 1 - \frac{(1-c)N_t}{K} \right) \right]
\]
where \( R \) and \( K \) are positive constants and \( c \) is the fraction harvested. Find \( N_t \) for \( t = 1, 2, \ldots, 20 \) when \( R = 1, K = 100 \), and \( N_0 = 50 \) for (a) \( c = 0.1 \), (b) \( c = 0.5 \), and (c) \( c = 0.9 \).

20. Harvesting Model Let \( N_t \) denote the population size at time \( t \), and assume that
\[
N_{t+1} = (1 - c) N_t \exp \left[ R \left( 1 - \frac{(1-c)N_t}{K} \right) \right]
\]
where \( R \) and \( K \) are positive constants and \( c \) is the fraction harvested. Find \( N_t \) for \( t = 1, 2, \ldots, 20 \) when \( R = 3, K = 100 \), and \( N_0 = 50 \) for (a) \( c = 0.1 \), (b) \( c = 0.5 \), and (c) \( c = 0.9 \).