Solution The function $f(x) = \frac{x^2-16}{x-4}$ is a rational function, but since $\lim_{x\to 4}(x-4) = 0$, we cannot use Rule 4. Instead, we need to simplify $f(x)$ first:

$$\lim_{x\to 4} \frac{x^2-16}{x-4} = \lim_{x\to 4} \frac{(x-4)(x+4)}{x-4}$$

Because $x \neq 4$, we can cancel $x-4$ in the numerator and denominator, which yields

$$\lim_{x\to 4}(x+4) = 8$$

where we used the fact that $x+4$ is a polynomial in computing the limit. ■

Section 3.1 Problems

■ 3.1.1

In Problems 1–32, use a table or a graph to investigate each limit.

1. $\lim_{x\to 2}(x^2-4x+1)$
2. $\lim_{x\to 2} \frac{x^2+3}{x+2}$
3. $\lim_{x\to -1} \frac{2x}{1+x^2}$
4. $\lim_{s\to 2} s(s^2-4)$
5. $\lim_{x\to \pi} 3\cos\frac{x}{4}$
6. $\lim_{t\to \pi/9} \sin(3t)$
7. $\lim_{x\to \pi/2} 2\sec\frac{x}{3}$
8. $\lim_{x\to \pi/2} \tan\frac{x-\pi/2}{2}$
9. $\lim_{x\to -2} e^{-x^2/2}$
10. $\lim_{x\to 0} \frac{e^x+1}{2x+3}$
11. $\lim_{x\to 0} \ln(x+1)$
12. $\lim_{t\to e} \ln t^3$
13. $\lim_{x\to 3} \frac{x^2-16}{x-4}$
14. $\lim_{x\to 2} \frac{x^2-4}{x+2}$
15. $\lim_{x\to \pi/2} \sin(2x)$
16. $\lim_{x\to \pi/2} \cos(x-\pi)$
17. $\lim_{x\to 0} \frac{1}{1+x^2}$
18. $\lim_{x\to 0} \frac{1}{x^2-1}$
19. $\lim_{x\to 0^+} (1-e^{-x})$
20. $\lim_{x\to 0^-} (1+e^x)$
21. $\lim_{x\to 4^-} \frac{2}{x-4}$
22. $\lim_{x\to 3^+} \frac{1}{x-3}$
23. $\lim_{x\to 1^-} \frac{2}{1-x}$
24. $\lim_{x\to 2^+} \frac{4}{2-x}$
25. $\lim_{x\to 1^-} \frac{1}{1-x^2}$
26. $\lim_{x\to 2^+} \frac{2}{x^2-4}$
27. $\lim_{x\to 3} \frac{1}{(x-3)^2}$
28. $\lim_{x\to 0} \frac{1-x^2}{x^2}$
29. $\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}$
30. $\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x}$
31. $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2}$
32. $\lim_{x\to 0} \frac{\sqrt{2-x}-\sqrt{2}}{2x}$
33. Use a table and a graph to find out what happens to

$$f(x) = \frac{2}{x^2}$$

as $x \to \infty$. What happens as $x \to -\infty$? What happens as $x \to 0$?

34. Use a table and a graph to find out what happens to

$$f(x) = \frac{2x}{x-1}$$

as $x \to \infty$. What happens as $x \to -\infty$? What happens as $x \to 1$?

35. Use a graphing calculator to investigate

$$\lim_{x\to 1} \sin\frac{1}{x-1}$$

36. Use a graphing calculator to investigate

$$\lim_{x\to 0} \cos\frac{1}{x}$$

■ 3.1.2

In Problems 37–54, use the limit laws to evaluate each limit.

37. $\lim_{x\to -1} (x^3+7x-1)$
38. $\lim_{x\to 2}(3x^4-2x+1)$
39. $\lim_{x\to -5} (4+2x^2)$
40. $\lim_{x\to 2}(8x^3-2x+4)$
41. $\lim_{x\to 3} \left(2x^2-\frac{1}{x}\right)$
42. $\lim_{x\to -2} \left(\frac{x^2}{2}-\frac{2}{x^2}\right)$
43. $\lim_{x\to -3} \frac{x^3-20}{x+1}$
44. $\lim_{x\to 1} \frac{x^3-1}{x+2}$
45. $\lim_{x\to 3} \frac{3x^2+1}{2x-3}$
46. $\lim_{x\to -2} \frac{1+x}{1-x}$
47. $\lim_{x\to 1} \frac{1-x^2}{1-x}$
48. $\lim_{u\to 3} \frac{9-u^2}{3-u}$
49. $\lim_{x\to 3} \frac{x^2-2x-3}{x-3}$
50. $\lim_{x\to 1} \frac{(x-1)^2}{x^2-1}$
51. $\lim_{x\to 2} \frac{2-x}{x^2-4}$
52. $\lim_{x\to -4} \frac{x+4}{16-x^2}$
53. $\lim_{x\to -2} \frac{2x^2+3x-2}{x+2}$
54. $\lim_{x\to 1/2} \frac{1-x-2x^2}{1-2x}$

EXAMPLE 9 Find

$$\lim_{x\to 1}\sqrt{2x^3-1}$$

Solution The function $f(x)=\sqrt{2x^3-1}$ is continuous at $x=1$. Thus,

$$\lim_{x\to 1}\sqrt{2x^3-1}=\sqrt{(2)(1)^3-1}=\sqrt{1}=1$$

EXAMPLE 10 Find

$$\lim_{x\to 0}e^{x-1}$$

Solution The function $f(x)=e^{x-1}$ is continuous at $x=0$. Therefore,

$$\lim_{x\to 0}e^{x-1}=e^{0-1}=e^{-1}$$

We conclude this section by calculating the limit of the expression in Example of Section 3.1.

EXAMPLE 11 Find

$$\lim_{x\to 0}\frac{\sqrt{x^2+16}-4}{x^2}$$

Solution We cannot apply Rule 4 of Section 3.1, since $f(x)=(\sqrt{x^2+16}-4)/x^2$ is not defin for $x=0$. (If we plug in 0, we get the expression 0/0.) We use a trick that will allo us to find the limit: We rationalize the numerator. For $x\neq 0$, we find that

$$\begin{aligned}\frac{\sqrt{x^2+16}-4}{x^2}&=\frac{(\sqrt{x^2+16}-4)}{x^2}\frac{(\sqrt{x^2+16}+4)}{(\sqrt{x^2+16}+4)}\\&=\frac{x^2+16-16}{x^2(\sqrt{x^2+16}+4)}=\frac{x^2}{x^2(\sqrt{x^2+16}+4)}\\&=\frac{1}{\sqrt{x^2+16}+4}\end{aligned}$$

Note that we are allowed to divide by x^2 in the last step, since we are assumi that $x\neq 0$. We can now apply Rule 4 to $1/(\sqrt{x^2+16}+4)$. When we do, we obtai

$$\lim_{x\to 0}\frac{\sqrt{x^2+16}-4}{x^2}=\lim_{x\to 0}\frac{1}{\sqrt{x^2+16}+4}=\frac{1}{8}=0.125$$

as we saw in Example 9 of Section 3.1. In Chapter 5, we will learn another meth for finding the limit of expressions of the form 0/0.

Section 3.2 Problems

■ 3.2.1

In Problems 1–4, show that each function is continuous at the given value.

1. $f(x)=2x,\ c=1/2$
2. $f(x)=-x,\ c=1$
3. $f(x)=x^3-2x+1,\ c=2$
4. $f(x)=x^2+1,\ c=-1$
5. Show that

$$f(x)=\begin{cases}\dfrac{x^2-x-2}{x-2} & \text{if } x\neq 2\\ 3 & \text{if } x=2\end{cases}$$

is continuous at $x=2$.

6. Show that

$$f(x)=\begin{cases}\dfrac{2x^2+x-6}{x+2} & \text{if } x\neq -2\\ -7 & \text{if } x=-2\end{cases}$$

is continuous at $x=-2$.

7. Let

$$f(x)=\begin{cases}\dfrac{x^2-9}{x-3} & \text{if } x\neq 3\end{cases}$$

Which value must you assign to a so that $f(x)$ is continuous at $x=3$?

8. Let

$$f(x)=\begin{cases}\dfrac{x^2+x-2}{x-1} & \text{if } x\neq 1\\ a & \text{if } x=1\end{cases}$$

Which value must you assign to a so that $f(x)$ is continuous at $x=1$?

In Problems 9–12, determine at which points $f(x)$ is discontinuous.

9. $f(x)=\dfrac{1}{x-3}$

10. $f(x)=\dfrac{1}{x^2-1}$

11. $f(x)=\begin{cases}\dfrac{x^2-3x+2}{x-2} & \text{if } x\neq 1\\ 1 & \text{if } x=1\end{cases}$

12. $f(x)=\begin{cases}x^2-1 & \text{if } x\leq 0\\ x & \text{if } x>0\end{cases}$

13. Show that the floor function $f(x)=\lfloor x\rfloor$ is continuous at $x=5/2$ but discontinuous at $x=3$.

14. Show that the floor function $f(x)=\lfloor x\rfloor$ is continuous from the right at $x=2$.

■ 3.2.2

In Problems 15–24, find the values of $x\in\mathbf{R}$ for which the given functions are continuous.

15. $f(x)=3x^4-x^2+4$

16. $f(x)=\sqrt{x^2-1}$

17. $f(x)=\dfrac{x^2+1}{x-1}$

18. $f(x)=\cos(2x)$

19. $f(x)=e^{-|x|}$

20. $f(x)=\ln(x-2)$

21. $f(x)=\ln\dfrac{x}{x+1}$

22. $f(x)=\exp[-\sqrt{x-1}]$

23. $f(x)=\tan(2\pi x)$

24. $f(x)=\sin\left(\dfrac{2x}{3+x}\right)$

25. Let

$$f(x)=\begin{cases}x^2+2 & \text{for } x\leq 0\\ x+c & \text{for } x>0\end{cases}$$

(a) Graph $f(x)$ when $c=1$, and determine whether $f(x)$ is continuous for this choice of c.

(b) How must you choose c so that $f(x)$ is continuous for all $x\in(-\infty,\infty)$?

26. Let

$$f(x)=\begin{cases}\dfrac{1}{x} & \text{for } x\geq 1\\ 2x+c & \text{for } x<1\end{cases}$$

(a) Graph $f(x)$ when $c=0$, and determine whether $f(x)$ is continuous for this choice of c.

(b) How must you choose c so that $f(x)$ is continuous for all $x\in(-\infty,\infty)$?

27. **(a)** Show that

$$f(x)=\sqrt{x-1},\quad x\geq 1$$

is continuous from the right at $x=1$.

(b) Graph $f(x)$.

(c) Does it make sense to look at continuity from the left at $x=1$?

28. **(a)** Show that

$$f(x)=\sqrt{x^2-4},\quad |x|\geq 2$$

is continuous from the right at $x=2$ and continuous from the left at $x=-2$.

(b) Graph $f(x)$.

(c) Does it make sense to look at continuity from the left at $x=2$ and at continuity from the right at $x=-2$?

In Problems 29–48, find the limits.

29. $\lim_{x\to\pi/3}\sin\left(\dfrac{x}{2}\right)$

30. $\lim_{x\to-\pi/2}\cos(2x)$

31. $\lim_{x\to\pi/2}\dfrac{\cos^2 x}{1-\sin^2 x}$

32. $\lim_{x\to-\pi/2}\dfrac{1+\tan^2 x}{\sec^2 x}$

33. $\lim_{x\to-1}\sqrt{4+5x^4}$

34. $\lim_{x\to-2}\sqrt{6+x}$

35. $\lim_{x\to-1}\sqrt{x^2+2x+2}$

36. $\lim_{x\to1}\sqrt{x^3+4x-1}$

37. $\lim_{x\to0}e^{-x^2/3}$

38. $\lim_{x\to0}e^{3x+2}$

39. $\lim_{x\to3}e^{x^2-9}$

40. $\lim_{x\to-1}e^{x^2/2-1}$

41. $\lim_{x\to0}\dfrac{e^{2x}-1}{e^x-1}$

42. $\lim_{x\to0}\dfrac{e^{-x}-e^x}{e^{-x}+1}$

43. $\lim_{x\to-2}\dfrac{1}{\sqrt{5x^2-4}}$

44. $\lim_{x\to1}\dfrac{1}{\sqrt{3-2x^2}}$

45. $\lim_{x\to0}\dfrac{\sqrt{x^2+9}-3}{x^2}$

46. $\lim_{x\to0}\dfrac{5-\sqrt{25+x^2}}{2x^2}$

47. $\lim_{x\to0}\ln(1-x)$

48. $\lim_{x\to1}\ln[e^x\cos(x-1)]$

■ 3.3 Limits at Infinity

The limit laws discussed in Subsection 3.1.2 also hold as x tends to ∞ (or $-\infty$).

EXAMPLE 1 Find

$$\lim_{x\to\infty}\frac{x}{x+1}$$

Section 3.3 Problems

Evaluate the limits in Problems 1–24.

1. $\lim_{x\to\infty} \frac{2x^2-3x+5}{x^4-2x+1}$
2. $\lim_{x\to\infty} \frac{x^2+3}{5x^2-2x+1}$
3. $\lim_{x\to-\infty} \frac{x^3+3}{x-2}$
4. $\lim_{x\to-\infty} \frac{2x-1}{3-4x}$
5. $\lim_{x\to\infty} \frac{1-x^3+2x^4}{2x^2+x^4}$
6. $\lim_{x\to\infty} \frac{1-5x^3}{1+3x^4}$
7. $\lim_{x\to\infty} \frac{x^2-2}{2x+1}$
8. $\lim_{x\to-\infty} \frac{3-x^2}{1-2x^2}$
9. $\lim_{x\to-\infty} \frac{x^2-3x+1}{4-x}$
10. $\lim_{x\to-\infty} \frac{1-x^3}{2+x}$
11. $\lim_{x\to-\infty} \frac{2+x^2}{1-x^2}$
12. $\lim_{x\to-\infty} \frac{2x+x^2}{3x+1}$
13. $\lim_{x\to\infty} \frac{4}{1+e^{-2x}}$
14. $\lim_{x\to\infty} \frac{e^{-x}}{1-e^{-x}}$
15. $\lim_{x\to\infty} \frac{2e^x}{e^x+3}$
16. $\lim_{x\to\infty} \frac{e^x}{2-e^x}$
17. $\lim_{x\to-\infty} \exp[x]$
18. $\lim_{x\to\infty} \exp[-\ln x]$
19. $\lim_{x\to\infty} \frac{3e^{2x}}{2e^{2x}-e^x}$
20. $\lim_{x\to\infty} \frac{3e^{2x}}{2e^{2x}-e^{3x}}$
21. $\lim_{x\to\infty} \frac{3}{2+e^{-x}}$
22. $\lim_{x\to-\infty} \frac{4}{1+e^{-x}}$
23. $\lim_{x\to-\infty} \frac{e^x}{1+x}$
24. $\lim_{x\to\infty} \frac{2}{e^x(1+x)}$

25. In Section 1.2.3, Example 6, we introduced the Monod growth function

$$r(N) = a\frac{N}{k+N}, \quad N \geq 0$$

Find $\lim_{N\to\infty} r(N)$.

26. In Problem 86 of Section 1.3, we discussed the Michaelis–Menten equation, which describes the initial velocity of an enzymatic reaction (v_0) as a function of the substrate concentration (s_0). The equation was given by

$$v_0 = \frac{v_{\max} s_0}{s_0 + K_m}$$

Find $\lim_{s_0\to\infty} v_0$.

27. Suppose the size of a population at time t is given by

$$N(t) = \frac{500t}{3+t}, \quad t \geq 0$$

(a) Use a graphing calculator to sketch the graph of $N(t)$.
(b) Determine the size of the population as $t \to \infty$. We call this the **limiting population size**.
(c) Show that, at time $t = 3$, the size of the population is half its limiting size.

28. **Logistic Growth** Suppose that the size of a population at time t is given by

$$N(t) = \frac{100}{1+9e^{-t}}$$

for $t \geq 0$.
(a) Use a graphing calculator to sketch the graph of $N(t)$.
(b) Determine the size of the population as $t \to \infty$, using the basic rules for limits. Compare your answer with the graph that you sketched in (a).

29. **Logistic Growth** Suppose that the size of a population at time t is given by

$$N(t) = \frac{50}{1+3e^{-t}}$$

for $t \geq 0$.
(a) Use a graphing calculator to sketch the graph of $N(t)$.
(b) Determine the size of the population as $t \to \infty$, using the basic rules for limits. Compare your answer with the graph that you sketched in (a).

3.4 The Sandwich Theorem and Some Trigonometric Limits

What happens during bungee jumping? The jumper is tied to an elastic rope, jumps off a bridge, and experiences damped oscillations until she comes to rest and will be hauled in to safety. The trajectory over time might resemble the function (Figure 3.17)

$$g(x) = e^{-x}\cos(10x), \quad x \geq 0$$

We suspect from the graph that

$$\lim_{x\to\infty} e^{-x}\cos(10x) = 0$$

If we wanted to calculate this limit, we would quickly see that none of the rules we have learned so far apply. Although $\lim_{x\to\infty} e^{-x} = 0$, we find that $\lim_{x\to\infty}\cos(10x)$ does not exist: The function $\cos(10x)$ oscillates between -1 and 1. Still, this property allows us to sandwich the function $g(x) = e^{-x}\cos(10x)$ between $f(x) = -e^{-x}$ and $h(x) = e^{-x}$. To do so, we note that from

$$-1 \leq \cos(10x) \leq 1$$

it follows that

$$-e^{-x} \leq e^{-x}\cos(10x) \leq e^{-x}$$

EXAMPLE 3 Find the following limits.

(a) $\lim_{x\to 0} \frac{\sin 3x}{5x}$ (b) $\lim_{x\to 0} \frac{\sin^2 x}{x^2}$ (c) $\lim_{x\to 0} \frac{\sec x - 1}{x \sec x}$

Solution (a) We cannot apply the first trigonometric limit directly. The trick is to substitu $z = 3x$ and observe that $z \to 0$ as $x \to 0$. Then

$$\lim_{x\to 0} \frac{\sin 3x}{5x} = \lim_{z\to 0} \frac{\sin z}{5z/3} = \frac{3}{5} \lim_{z\to 0} \frac{\sin z}{z} = \frac{3}{5}$$

(b) We note that

$$\lim_{x\to 0} \frac{\sin^2 x}{x^2} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = 1$$

Here, we used the fact that the limit of a product is the product of the limits, provid that the individual limits exist.

(c) We first write $\sec x = 1/\cos x$ and then multiply both numerator a denominator by $\cos x$:

$$\lim_{x\to 0} \frac{\sec x - 1}{x \sec x} = \lim_{x\to 0} \frac{\frac{1}{\cos x} - 1}{\frac{x}{\cos x}}$$

$$= \lim_{x\to 0} \frac{\left(\frac{1}{\cos x} - 1\right)\cos x}{\frac{x}{\cos x}\cos x} = \lim_{x\to 0} \frac{1 - \cos x}{x} = 0$$

Section 3.4 Problems

1. Let

$$f(x) = x^2 \cos\frac{1}{x}, \quad x \neq 0$$

(a) Use a graphing calculator to sketch the graph of $y = f(x)$.

(b) Show that

$$-x^2 \le x^2 \cos\frac{1}{x} \le x^2$$

holds for $x \neq 0$.

(c) Use your result in (b) and the sandwich theorem to show that

$$\lim_{x\to 0} x^2 \cos\frac{1}{x} = 0$$

2. Let

$$f(x) = x \cos\frac{1}{x}, \quad x \neq 0$$

(a) Use a graphing calculator to sketch the graph of $y = f(x)$.

(b) Use the sandwich theorem to show that

$$\lim_{x\to 0} x \cos\frac{1}{x} = 0$$

3. Let

$$f(x) = \frac{\ln x}{x}, \quad x > 0$$

(a) Use a graphing calculator to graph $y = f(x)$.

(b) Use a graphing calculator to investigate the values of x fo which

$$\frac{1}{x} \le \frac{\ln x}{x} \le \frac{1}{\sqrt{x}}$$

holds.

(c) Use your result in (b) to explain why the following is true:

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0$$

4. Let

$$f(x) = \frac{\sin x}{x}, \quad x > 0$$

(a) Use a graphing calculator to graph $y = f(x)$.

(b) Explain why you cannot use the basic rules for finding limits to compute

$$\lim_{x\to\infty} \frac{\sin x}{x}$$

(c) Show that

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

holds for $x > 0$, and use the sandwich theorem to compute

$$\lim_{x\to\infty} \frac{\sin x}{x}$$

In Problems 5–20, evaluate the trigonometric limits.

5. $\lim_{x\to 0} \frac{\sin(2x)}{2x}$

6. $\lim_{x\to 0} \frac{\sin(2x)}{3x}$

7. $\lim_{x\to 0} \frac{\sin(5x)}{x}$

8. $\lim_{x\to 0} \frac{\sin x}{-x}$

9. $\lim_{x\to 0} \frac{\sin(\pi x)}{x}$

10. $\lim_{x\to 0} \frac{\sin(-\pi x/2)}{2x}$

11. $\lim_{x\to 0} \frac{\sin(\pi x)}{\sqrt{x}}$

12. $\lim_{x\to 0} \frac{\sin^2 x}{x}$

13. $\lim_{x\to 0} \frac{\sin x \cos x}{x(1-x)}$

14. $\lim_{x\to 0} \frac{1-\cos^2 x}{x^2}$

15. $\lim_{x\to 0} \frac{1-\cos x}{2x}$

16. $\lim_{x\to 0} \frac{1-\cos(2x)}{3x}$

17. $\lim_{x\to 0} \frac{1-\cos(5x)}{2x}$

18. $\lim_{x\to 0} \frac{1-\cos(x/2)}{x}$

19. $\lim_{x\to 0} \frac{\sin x(1-\cos x)}{x^2}$

20. $\lim_{x\to 0} \frac{\csc x - \cot x}{x \csc x}$

21. **(a)** Use a graphing calculator to sketch the graph of

$$f(x) = e^{ax} \sin x, \quad x \geq 0$$

for $a = -0.1, -0.01, 0, 0.01$, and 0.1.

(b) Which part of the function $f(x)$ produces the oscillations that you see in the graphs sketched in (a)?

(c) Describe in words the effect that the value of a has on the shape of the graph of $f(x)$.

(d) Graph $f(x) = e^{ax} \sin x$, $g(x) = -e^{ax}$, and $h(x) = e^{ax}$ together in one coordinate system for **(i)** $a = 0.1$ and **(ii)** $a = -0.1$. [Use separate coordinate systems for (i) and (ii).] Explain what you see in each case. Show that

$$-e^{ax} \leq e^{ax} \sin x \leq e^{ax}$$

Use this pair of inequalities to determine the values of a for which

$$\lim_{x\to\infty} f(x)$$

exists, and find the limiting value.

3.5 Properties of Continuous Functions

3.5.1 The Intermediate-Value Theorem

As you hike up a mountain, the temperature decreases with increasing elevation. Suppose the temperature at the bottom of the mountain is 70°F and the temperature at the top of the mountain is 40°F. How do you know that at some time during your hike you must have crossed a point where the temperature was exactly 50°F? Your answer will probably be something like the following: "To go from 70°F to 40°F, I must have passed through 50°F, since 50°F is between 40°F and 70°F and the temperature changed continuously as I hiked up the mountain." This statement represents the content of the intermediate-value theorem.

The Intermediate-Value Theorem Suppose that f is continuous on the closed interval $[a, b]$. If L is any real number with $f(a) < L < f(b)$ or $f(b) < L < f(a)$, then there exists at least one number c on the open interval (a, b) such that $f(c) = L$.

We will not prove this theorem, but Figure 3.21 should convince you that it is true. In the figure, f is continuous and defined on the closed interval $[a, b]$ with $f(a) < L < f(b)$. Therefore, the graph of $f(x)$ must intersect the line $y = L$ at least once on the open interval (a, b).

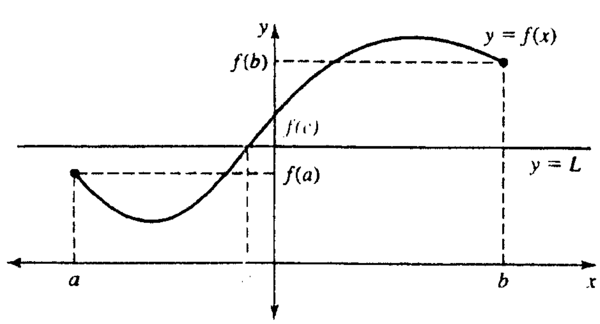

Figure 3.21 The intermediate-value theorem.

as we saw. The bisection method is fairly slow when we need high accuracy. For instance, to reduce the length of the interval to 10^{-6}, we would need at least 22 steps, since

$$3 \cdot \left(\frac{1}{2}\right)^{21} > 10^{-6} > 3 \cdot \left(\frac{1}{2}\right)^{22}$$

In Section 5.7, we will learn a faster method.

Figure 3.23 shows the graph of $f(x) = x^5 - 7x^2 + 3$. We see that the graph intersects the x-axis three times. We found an approximation of the leftmost root of the equation $x^5 - 7x^2 + 3$. If we had used another starting interval—say, $[1, 2]$—we would have located an approximation of the rightmost root of the equation. ■

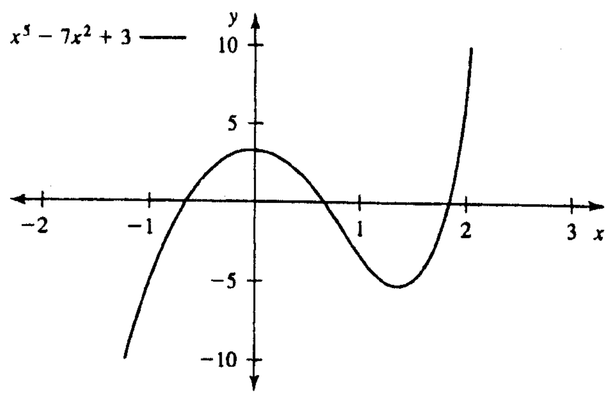

Figure 3.23 The graph of $f(x) = x^5 - 7x^2 + 3$.

■ 3.5.2 A Final Remark on Continuous Functions

Many functions in biology are in fact discontinuous. For example, if we measure the size of a population over time, we find that it takes on discrete values only (namely, nonnegative integers) and therefore changes discontinuously. However, if the population size is sufficiently large, an increase or decrease by 1 changes the population size so slightly that it might be justified to approximate it by a continuous function. For example, if we measure the number of bacteria, in millions, in a petri dish, then the number 2.1 would correspond to 2,100,000 bacteria. An increase by 1 results in 2,100,001 bacteria, or, if we measure the size in millions, in 2.100001, an increase of 10^{-6}.

Section 3.5 Problems

■ 3.5.1, 3.5.2

1. Let

$$f(x) = x^2 - 1, \quad 0 \le x \le 2$$

(a) Graph $y = f(x)$ for $0 \le x \le 2$.

(b) Show that

$$f(0) < 0 < f(2)$$

and use the intermediate-value theorem to conclude that there exists a number $c \in (0, 2)$ such that $f(c) = 0$.

2. Let

$$f(x) = x^3 - 2x + 3, \quad -3 \le x \le -1$$

(a) Graph $y = f(x)$ for $-3 \le x \le -1$.

(b) Use the intermediate-value theorem to conclude that

$$x^3 - 2x + 3 = 0$$

has a solution in $(-3, -1)$.

3. Let

$$f(x) = \sqrt{x^2 + 2}, \quad 1 \le x \le 2$$

(a) Graph $y = f(x)$ for $1 \le x \le 2$.

(b) Use the intermediate-value theorem to conclude that

$$\sqrt{x^2 + 2} = 2$$

has a solution in $(1, 2)$.

4. Let

$$f(x) = \sin x - x, \quad -1 \le x \le 1$$

(a) Graph $y = f(x)$ for $-1 \le x \le 1$.

(b) Use the intermediate-value theorem to conclude that

$$\sin x = x$$

has a solution in $(-1, 1)$.

5. Use the intermediate-value theorem to show that

$$e^{-x} = x$$

has a solution in (0, 1).

6. Use the intermediate-value theorem to show that

$$\cos x = x$$

has a solution in (0, 1).

7. Use the bisection method to find a solution of

$$e^{-x} = x$$

that is accurate to two decimal places.

8. Use the bisection method to find a solution of

$$\cos x = x$$

that is accurate to two decimal places.

9. **(a)** Use the bisection method to find a solution of $3x^3 - 4x^2 - x + 2 = 0$ that is accurate to two decimal places.
(b) Graph the function $f(x) = 3x^3 - 4x^2 - x + 2$.
(c) Which solution did you locate in (a)? Is it possible in this case to find the other solution by using the bisection method together with the intermediate-value theorem?

10. In Example 2, how many steps are required to guarantee that the approximate root is within 0.0001 of the true value of the root?

11. Suppose that the number of individuals in a population at time t is given by

$$N(t) = \frac{54t}{13+t}, \quad t \geq 0 \tag{3.7}$$

(a) Use a calculator to confirm that $N(10)$ is approximately 23.47826. Considering that the number of individuals in a population is an integer, how should you report your answer?
(b) Now suppose that $N(t)$ is given by the same function (3.7), but that the size of the population is measured in millions. How should you report the population size at time $t = 10$? Make some reasonable assumptions about the accuracy of a measurement for the size of such a large population.
(c) Discuss the use of continuous functions in both (a) and (b).

12. Suppose that the biomass of a population at time t is given by

$$B(t) = \frac{32.00t}{17.00+t}, \quad t \geq 0 \tag{3.8}$$

(a) Use a calculator to confirm that $B(10)$ is approximately 1.185185. Considering the function $B(t)$, how many significant figures should you report in your answer?
(b) Discuss the use of continuous functions in this problem.

13. Explain why a polynomial of degree 3 has at least one root.

14. Explain why a polynomial of degree n, where n is an odd number, has at least one root.

15. Explain why $y = x^2 - 4$ has at least two roots.

16. On the basis of the intermediate-value theorem, what can you say about the number of roots of a polynomial of even degree?

■ 3.6 A Formal Definition of Limits (Optional)

The ancient Greeks used limiting procedures to compute areas, such as the area of a circle, by the "method of exhaustion." In this method, a region was covered (or "exhausted") as closely as possible by triangles. Adding the areas of the triangles then yielded an approximation of the area of the region of interest. Newton and Leibniz, the inventors of calculus, were aware of the importance of taking limits in their development of the subject; however, they did not give a rigorous definition of the procedure. The French mathematician Augustin-Louis Cauchy (1789–1857) was the first to develop a rigorous definition of limits; the definition we will use goes back to the German mathematician Karl Weierstrass (1815–1897).

Before we write the formal definition, let's return to the informal one. In that definition, we stated that $\lim_{x\to c} f(x) = L$ means that the value of $f(x)$ can be made arbitrarily close to L whenever x is sufficiently close to c. But just how close is sufficient? Take Example 1 from Section 3.1: Suppose we wish to show that

$$\lim_{x\to 2} x^2 = 4$$

without using the continuity of $y = x^2$, which itself was based on $\lim_{x\to c} x = c$ [Equation (3.3)]. What would we have to do? We would need to show that x^2 can be made arbitrarily close to 4 for all values of x sufficiently close, but not equal, to 2. (In what follows, we will always exclude $x = 2$ from the discussion, since the value of x^2 at $x = 2$ is irrelevant in finding the limit.) Suppose we wish to make x^2 within 0.01 of 4; that is, we want $|x^2 - 4| < 0.01$. Does this inequality hold for all x sufficiently close, but not equal, to 2? We begin with

$$|x^2 - 4| < 0.01$$

Section 3.6 Problems

1. Find the values of x such that
$$|2x - 1| < 0.01$$

2. Find the values of x such that
$$|3x - 9| < 0.01$$

3. Find the values of x such that
$$|x^2 - 9| < 0.1$$

4. Find the values of x such that
$$|2\sqrt{x} - 5| < 0.1$$

5. Let
$$f(x) = 2x - 1,\ x \in \mathbf{R}$$
(a) Graph $y = f(x)$ for $-3 \le x \le 5$.
(b) For which values of x is $y = f(x)$ within 0.1 of 3? [*Hint*: Find values of x such that $|(2x - 1) - 3| < 0.1$.]
(c) Illustrate your result in (b) on the graph that you obtained in (a).

6. Let
$$f(x) = \sqrt{x}, \quad x \ge 0$$
(a) Graph $y = f(x)$ for $0 \le x \le 6$.
(b) For which values of x is $y = f(x)$ within 0.2 of 1? (*Hint*: Find values of x such that $|\sqrt{x} - 1| < 0.2$.)
(c) Illustrate your result in (b) on the graph that you obtained in (a).

7. Let
$$f(x) = \frac{1}{x}, \quad x > 0$$
(a) Graph $y = f(x)$ for $0 < x \le 4$.
(b) For which values of x is $y = f(x)$ greater than 4?
(c) Illustrate your result in (b) on the graph that you obtained in (a).

8. Let
$$f(x) = e^{-x}, \quad x \ge 0$$
(a) Graph $y = f(x)$ for $0 \le x \le 6$.
(b) For which values of x is $y = f(x)$ less than 0.1?
(c) Illustrate your result in (b) on the graph that you obtained in (a).

In Problems 9–22, use the formal definition of limits to prove each statement.

9. $\lim_{x\to 2}(2x - 1) = 3$
10. $\lim_{x\to 0} x^2 = 0$
11. $\lim_{x\to 0} x^5 = 0$
12. $\lim_{x\to 1} \frac{1}{x} = 1$
13. $\lim_{x\to 0} \frac{4}{x^2} = \infty$
14. $\lim_{x\to 0} \frac{-2}{x^2} = -\infty$
15. $\lim_{x\to 0} \frac{1}{x^4} = \infty$
16. $\lim_{x\to 3} \frac{1}{(x-3)^2} = \infty$
17. $\lim_{x\to\infty} \frac{2}{x^2} = 0$
18. $\lim_{x\to\infty} e^{-x} = 0$
19. $\lim_{x\to\infty} \frac{x}{x+1} = 1$
20. $\lim_{x\to-\infty} \frac{x}{x+1} = 1$
21. $\lim_{x\to c}(mx) = mc$, where m is a constant
22. $\lim_{x\to c}(mx + b) = mc + b$, where m and b are constants

Chapter 3 Key Terms

Discuss the following definitions and concepts:

1. Limit of $f(x)$ as x approaches c
2. One-sided limits
3. Infinite limits
4. Divergence by oscillations
5. Convergence
6. Divergence
7. Limit laws
8. Continuity
9. One-sided continuity
10. Continuous function
11. Removable discontinuity
12. Sandwich theorem
13. Trigonometric limits
14. Intermediate-value theorem
15. Bisection method
16. ϵ–δ definition of limits

Chapter 3 Review Problems

In Problems 1–4, determine where each function is continuous. Investigate the behavior as $x \to \pm\infty$. Use a graphing calculator to sketch the corresponding graphs.

1. $f(x) = e^{-|x|}$
2. $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$
3. $f(x) = \frac{2}{e^x + e^{-x}}$
4. $f(x) = \frac{1}{\sqrt{x^2 - 1}}$

5. Sketch the graph of a function that is discontinuous from the left and continuous from the right at $x = 1$.

6. Sketch the graph of a function $f(x)$ that is continuous on $[0, 2]$, except at $x = 1$, where $f(1) = 4$, $\lim_{x\to 1^-} f(x) = 2$, and $\lim_{x\to 1^-} f(x) = 3$.

7. Sketch the graph of a continuous function on $[0, \infty)$ with $f(0) = 0$ and $\lim_{x\to\infty} f(x) = 1$.

8. Sketch the graph of a continuous function on $(-\infty, \infty)$ with $f(0) = 1$, $f(x) \ge 0$ for all $x \in \mathbf{R}$, and $\lim_{x\to\pm\infty} f(x) = 0$.

9. Show that the floor function
$$f(x) = \lfloor x \rfloor$$
is continuous from the right, but discontinuous from the left at $x = -2$.

10. Suppose $f(x)$ is continuous on the interval $[1, 3]$. If $f(1) = 0$ and $f(3) = 2$, explain why there must be a number $c \in (1, 3)$ such that $f(c) = 1$.

11. **Population Size** Assume that the size of a population at time t is
$$N(t) = \frac{at}{k + t}, \quad t \ge 0$$

where a and k are positive constants. Suppose that the limiting population size is

$$\lim_{t\to\infty} N(t) = 1.24 \times 10^6$$

and that, at time $t = 5$, the population size is half the limiting population size. Use the preceding information to determine the constants a and k.

12. Population Size Suppose that

$$N(t) = 10 + 2e^{-0.3t}\sin t, \quad t \geq 0$$

describes the size of a population (in millions) at time t (measured in weeks).

(a) Use a graphing calculator to sketch the graph of $N(t)$, and describe in words what you see.

(b) Give lower and upper bounds on the size of the population; that is, find N_1 and N_2 such that, for all $t \geq 0$,

$$N_1 \leq N(t) \leq N_2$$

(c) Find $\lim_{t\to\infty} N(t)$. Interpret this expression.

13. Physiology Suppose that an organism reacts to a stimulus only when the stimulus exceeds a certain threshold. Assume that the stimulus is a function of time t and that it is given by

$$s(t) = \sin(\pi t), \quad t \geq 0$$

The organism reacts to the stimulus and shows a certain reaction when $s(t) \geq 1/2$. Define a function $g(t)$ such that $g(t) = 0$ when the organism shows no reaction at time t and $g(t) = 1$ when the organism shows the reaction.

(a) Plot $s(t)$ and $g(t)$ in the same coordinate system.

(b) Is $s(t)$ continuous? Is $g(t)$ continuous?

14. Tree Height The following function describes the height of a tree as a function of age:

$$f(x) = 132e^{-20/x}, \quad x \geq 0$$

Find $\lim_{x\to\infty} f(x)$.

15. Predator–Prey Model There are a number of mathematical models that describe predator–prey interactions. Typically, they share the feature that the number of prey eaten per predator increases with the density of the prey. In the simplest version, the number of encounters with prey per predator is proportional to the product of the total number of prey and the period over which the predators search for prey. That is, if we let N be the number of prey, P be the number of predators, T be the period available for searching, and N_e be the number of encounters with prey, then

$$\frac{N_e}{P} = aTN \tag{3.11}$$

where a is a positive constant. The quantity N_e/P is the number of prey encountered per predator.

(a) Set $f(N) = aTN$, and sketch the graph of $f(N)$ when $a = 0.1$ and $T = 2$ for $N \geq 0$.

(b) Predators usually spend some time eating the prey that they find. Therefore, not all of the time T can be used for searching. The actual searching time is reduced by the per-prey handling time T_h and can be written as

$$T - T_h\frac{N_e}{P}$$

Show that if $T - T_h\frac{N_e}{P}$ is substituted for T in (3.11), then

$$\frac{N_e}{P} = \frac{aTN}{1 + aT_hN} \tag{3.12}$$

Define

$$g(N) = \frac{aTN}{1 + aT_hN}$$

and graph $g(N)$ for $N \geq 0$ when $a = 0.1$, $T = 2$, and $T_h = 0.1$.

(c) Show that (3.12) reduces to (3.11) when $T_h = 0$.

(d) Find

$$\lim_{N\to\infty} \frac{N_e}{P}$$

in the cases when $T_h = 0$ and when $T_h > 0$. Explain, in words, the difference between the two cases.

16. Community Respiration Duarte and Agustí (1998) investigated the CO_2 balance of aquatic ecosystems. They related the community respiration rates (R) to the gross primary production rates (P) of aquatic ecosystems. (Both quantities were measured in the same units.) They made the following statement:

> Our results confirm the generality of earlier reports that the relation between community respiration rate and gross production is not linear. Community respiration is scaled as the approximate two-thirds power of gross production.

(a) Use the preceding quote to explain why

$$R = aP^b$$

can be used to describe the relationship between the community respiration rates (R) and the gross primary production rates (P). What value would you assign to b on the basis of their quote?

(b) Suppose that you obtained data on the gross production and respiration rates of a number of freshwater lakes. How would you display your data graphically to quickly convince an audience that the exponent b in the power equation relating R and P is indeed approximately 2/3? (*Hint*: Use an appropriate log transformation.)

(c) The ratio R/P for an ecosystem is important in assessing the global CO_2 budget. If respiration exceeds production (i.e., $R > P$), then the ecosystem acts as a carbon dioxide source, whereas if production exceeds respiration (i.e., $P > R$), then the ecosystem acts as a carbon dioxide sink. Assume now that the exponent in the power equation relating R and P is 2/3. Show that the ratio R/P, as a function of P, is continuous for $P > 0$. Furthermore, show that

$$\lim_{P\to 0+} \frac{R}{P} = \infty$$

and

$$\lim_{P\to\infty} \frac{R}{P} = 0$$

Use a graphing calculator to sketch the graph of the ratio R/P as a function of P for $P > 0$. (Experiment with the graphing calculator to see how the value of a affects the graph.)

(d) Use your results in (c) and the intermediate-value theorem to conclude that there exists a value P^* such that the ratio R/P at P^* is equal to 1. On the basis of your graph in (c), is there more than one such value P^*?

(e) Use your results in (d) to identify production rates P where the ratio $R/P > 1$ (i.e., where respiration exceeds production).

(f) Use your results in (a)–(e) to explain the following quote from Duarte and Agustí:

> Unproductive aquatic ecosystems ... tend to be heterotrophic ($R > P$), and act as carbon dioxide sources.

17. Hyperbolic functions are used in the sciences. We take a look at the following three examples: the hyperbolic sine, $\sinh x$; the hyperbolic cosine, $\cosh x$; and the hyperbolic tangent, $\tanh x$, defined respectively as

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad x \in \mathbf{R}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad x \in \mathbf{R}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad x \in \mathbf{R}$$

(a) Show that these three hyperbolic functions are continuous for all $x \in \mathbf{R}$. Use a graphing calculator to sketch the graphs of all three functions.

(b) Find

$$\lim_{x\to\infty} \sinh x \qquad \lim_{x\to-\infty} \sinh x$$

$$\lim_{x\to\infty} \cosh x \qquad \lim_{x\to-\infty} \cosh x$$

$$\lim_{x\to\infty} \tanh x \qquad \lim_{x\to-\infty} \tanh x$$

(c) Show that the two identities

$$\cosh^2 x - \sinh^2 x = 1$$

and

$$\tanh x = \frac{\sinh x}{\cosh x}$$

are valid.

(d) Show that $\sinh x$ and $\tanh x$ are odd functions and that $\cosh x$ is even.

(*Note*: It can be shown that if a flexible cable is suspended between two points at equal heights, the shape of the resulting curve is given by the hyperbolic cosine function. This curve is called a *catenary*.)