EXAMPLE 10 Assume that f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$, with $f(0) = 2$ and $f'(x) = 0$ for all $x \in (-1, 1)$. Find $f(x)$.

Solution Corollary 2 tells us that $f(x)$ is a constant. Since we know that $f(0) = 2$, we have $f(x) = 2$ for all $x \in [-1, 1]$. ■

Proof of Corollary 2 Let $x_1, x_2 \in (a, b), x_1 < x_2$. Then f satisfies the assumptions of the MVT on the closed interval $[x_1, x_2]$. Therefore, there exists a number $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

Since $f'(c) = 0$, it follows that $f(x_2) = f(x_1)$. Finally, because x_1, x_2 are arbitrary numbers from the interval (a, b), we conclude that f is constant. ■

EXAMPLE 11 Show that

$$\sin^2 x + \cos^2 x = 1 \quad \text{for all } x \in [0, 2\pi]$$

Solution This identity can be shown without calculus, but let's see what we get if we use Corollary 2. We define $f(x) = \sin^2 x + \cos^2 x, 0 \le x \le 2\pi$. Then $f(x)$ is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$, with

$$f'(x) = 2 \sin x \cos x - 2 \cos x \sin x = 0$$

Using Corollary 2 now, we conclude that $f(x)$ is equal to a constant on $[0, 2\pi]$. To find the constant, we need only evaluate $f(x)$ at one point in the interval, say, $x = 0$. We find that

$$f(0) = \sin^2 0 + \cos^2 0 = 1$$

This proves the identity. ■

Section 5.1 Problems

■ 5.1.1

In Problems 1–8, each function is continuous and defined on a closed interval. It therefore satisfies the assumptions of the extreme-value theorem. With the help of a graphing calculator, graph each function and locate its global extrema. (Note that a function may assume a global extremum at more than one point.)

1. $f(x) = 2x - 1, 0 \le x \le 1$
2. $f(x) = -x^2 + 1, -1 \le x \le 1$
3. $f(x) = \sin(2x), 0 \le x \le \pi$
4. $f(x) = \cos \frac{x}{2}, 0 \le x \le 2\pi$
5. $f(x) = |x|, -1 \le x \le 1$
6. $f(x) = (x - 1)^2(x + 2), -2 \le x \le 2$
7. $f(x) = e^{-|x|}, -1 \le x \le 1$
8. $f(x) = \ln(x + 1), 0 \le x \le 2$
9. Sketch the graph of a function that is continuous on the closed interval $[0, 3]$ and has a global maximum at the left endpoint and a global minimum at the right endpoint.
10. Sketch the graph of a function that is continuous on the closed interval $[-2, 1]$ and has a global maximum and a global minimum in the interior of the domain of the function.
11. Sketch the graph of a function that is continuous on the open interval $(0, 2)$ and has neither a global maximum nor a global minimum in its domain.
12. Sketch the graph of a function that is continuous on the closed interval $[1, 4]$, except at $x = 2$, and has neither a global maximum nor a global minimum in its domain.

■ 5.1.2

In Problems 13–18, use a graphing calculator to determine all local and global extrema of the functions on their respective domains.

13. $f(x) = 3 - x, x \in [-1, 3)$
14. $f(x) = 5 + 2x, x \in (-2, 1)$
15. $f(x) = x^2 - 2, x \in [-1, 1]$
16. $f(x) = (x - 2)^2, x \in [0, 3]$
17. $f(x) = -x^2 + 1, x \in [-2, 1]$
18. $f(x) = x^2 - x, x \in [0, 1]$

In Problems 19–26, find c such that $f'(c) = 0$ and determine whether $f(x)$ has a local extremum at $x = c$.

19. $f(x) = x^2$
20. $f(x) = (x - 4)^2$
21. $f(x) = -x^2$
22. $f(x) = -(x + 3)^2$
23. $f(x) = x^3$
24. $f(x) = x^5$
25. $f(x) = (x + 1)^3$
26. $f(x) = -(x - 3)^5$
27. Show that $f(x) = |x|$ has a local minimum at $x = 0$ but $f(x)$ is not differentiable at $x = 0$.
28. Show that $f(x) = |x - 1|$ has a local minimum at $x = 1$ but $f(x)$ is not differentiable at $x = 1$.
29. Show that $f(x) = |x^2 - 1|$ has local minima at $x = 1$ and $x = -1$ but $f(x)$ is not differentiable at $x = 1$ or $x = -1$.

30. Show that $f(x) = -|x^2 - 4|$ has local maxima at $x = 2$ and $x = -2$ but $f(x)$ is not differentiable at $x = 2$ or $x = -2$.

31. Graph

$$f(x) = \left|1 - |x|\right|, \quad -1 \le x \le 2$$

and determine all local and global extrema on $[-1, 2]$.

32. Graph

$$f(x) = -\left||x| - 2\right|, \quad -3 \le x \le 3$$

and determine all local and global extrema on $[-3, 3]$.

33. Suppose the size of a population at time t is $N(t)$ and its growth rate is given by the logistic growth function

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad t \ge 0$$

where r and K are positive constants.

(a) Graph the growth rate $\frac{dN}{dt}$ as a function of N for $r = 2$ and $K = 100$, and find the population size for which the growth rate is maximal.

(b) Show that $f(N) = rN(1 - N/K)$, $N \ge 0$, is differentiable for $N > 0$, and compute $f'(N)$.

(c) Show that $f'(N) = 0$ for the value of N that you determined in (a) when $r = 2$ and $K = 100$.

34. Suppose that the size of a population at time t is $N(t)$ and its growth rate is given by the logistic growth function

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad t \ge 0$$

where r and K are positive constants. The per capita growth rate is defined by

$$g(N) = \frac{1}{N}\frac{dN}{dt}$$

(a) Show that

$$g(N) = r\left(1 - \frac{N}{K}\right)$$

(b) Graph $g(N)$ as a function of N for $N \ge 0$ when $r = 2$ and $K = 100$, and find the population size for which the per capita growth rate is maximal.

■ 5.1.3

35. Suppose $f(x) = x^2$, $x \in [0, 2]$.

(a) Find the slope of the secant line connecting the points (0, 0) and (2, 4).

(b) Find a number $c \in (0, 2)$ such that $f'(c)$ is equal to the slope of the secant line you computed in (a), and explain why such a number must exist in (0, 2).

36. Suppose $f(x) = 1/x$, $x \in [1, 2]$.

(a) Find the slope of the secant line connecting the points (1, 1) and (2, 1/2).

(b) Find a number $c \in (1, 2)$ such that $f'(c)$ is equal to the slope of the secant line you computed in (a), and explain why such a number must exist in (1, 2).

37. Suppose that $f(x) = x^2$, $x \in [-1, 1]$.

(a) Find the slope of the secant line connecting the points (−1, 1) and (1, 1).

(b) Find a number $c \in (-1, 1)$ such that $f'(c)$ is equal to the slope of the secant line you computed in (a), and explain why such a number must exist in (−1, 1).

38. Suppose that $f(x) = x^2 - x - 2$, $x \in [-1, 2]$.

(a) Find the slope of the secant line connecting the points (−1, 0) and (2, 0).

(b) Find a number $c \in (-1, 2)$ such that $f'(c)$ is equal to the slope of the secant line you computed in (a), and explain why such a number must exist in (−1, 2).

39. Let $f(x) = x(1 - x)$. Use the MVT to find an interval that contains a number c such that $f'(c) = 0$.

40. Let $f(x) = 1/(1 + x^2)$. Use the MVT to find an interval that contains a number c such that $f'(c) = 0$.

41. Suppose that $f(x) = -x^2 + 2$. Explain why there exists a point c in the interval $(-1, 2)$ such that $f'(c) = -1$.

42. Suppose that $f(x) = x^3$. Explain why there exists a point c in the interval $(-1, 1)$ such that $f'(c) = 1$.

43. Sketch the graph of a function $f(x)$ that is continuous on the closed interval [0, 1] and differentiable on the open interval (0, 1) such that there exists exactly one point $(c, f(c))$ on the graph at which the slope of the tangent line is equal to the slope of the secant line connecting the points $(0, f(0))$ and $(1, f(1))$. Why can you be sure that there is such a point?

44. Sketch the graph of a function $f(x)$ that is continuous on the closed interval [0, 1] and differentiable on the open interval (0, 1) such that there exist exactly two points $(c_1, f(c_1))$ and $(c_2, f(c_2))$ on the graph at which the slope of the tangent lines is equal to the slope of the secant line connecting the points $(0, f(0))$ and $(1, f(1))$. Why can you be sure that there is at least one such point?

45. Suppose that $f(x) = x^2$, $x \in [a, b]$.

(a) Compute the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$.

(b) Find the point $c \in (a, b)$ such that the slope of the tangent line to the graph of f at $(c, f(c))$ is equal to the slope of the secant line determined in (a). How do you know that such a point exists? Show that c is the midpoint of the interval (a, b); that is, show that $c = (a + b)/2$.

46. Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Show that if $f(a) < f(b)$, then f' is positive at some point between a and b.

47. Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Assume further that $f(a) = f(b) = 0$ but f is not constant on $[a, b]$. Explain why there must be a point $c_1 \in (a, b)$ with $f'(c_1) > 0$ and a point $c_2 \in (a, b)$ with $f'(c_2) < 0$.

48. A car moves in a straight line. At time t (measured in seconds), its position (measured in meters) is

$$s(t) = \frac{1}{10}t^2, \quad 0 \le t \le 10$$

(a) Find its average velocity between $t = 0$ and $t = 10$.

(b) Find its instantaneous velocity for $t \in (0, 10)$.

(c) At what time is the instantaneous velocity of the car equal to its average velocity?

49. A car moves in a straight line. At time t (measured in seconds), its position (measured in meters) is

$$s(t) = \frac{1}{100}t^3, \quad 0 \le t \le 5$$

(a) Find its average velocity between $t = 0$ and $t = 5$.

(b) Find its instantaneous velocity for $t \in (0, 5)$.

(c) At what time is the instantaneous velocity of the car equal to its average velocity?

50. Denote the population size at time t by $N(t)$, and assume that $N(0) = 50$ and $|dN/dt| \le 2$ for all $t \in [0, 5]$. What can you say about $N(5)$?

51. Denote the biomass at time t by $B(t)$, and assume that $B(0) = 3$ and $|dB/dt| \leq 1$ for all $t \in [0, 3]$. What can you say about $B(3)$?

52. Suppose that f is differentiable for all $x \in \mathbf{R}$ and, furthermore, that f satisfies $f(0) = 0$ and $1 \leq f'(x) \leq 2$ for all $x > 0$.
(a) Use Corollary 1 of the MVT to show that

$$x \leq f(x) \leq 2x$$

for all $x \geq 0$.
(b) Use your result in (a) to explain why $f(1)$ cannot be equal to 3.
(c) Find an upper and a lower bound for the value of $f(1)$.

53. Suppose that f is differentiable for all $x \in \mathbf{R}$ with $f(2) = 3$ and $f'(x) = 0$ for all $x \in \mathbf{R}$. Find $f(x)$.

54. Suppose that $f(x) = e^{-|x|}$, $x \in [-2, 2]$.
(a) Show that $f(-2) = f(2)$.
(b) Compute $f'(x)$, where defined.
(c) Show that there is no number $c \in (-2, 2)$ such that $f'(c) = 0$.
(d) Explain why your results in (a) and (c) do not contradict Rolle's theorem.
(e) Use a graphing calculator to sketch the graph of $f(x)$.

55. Use Corollary 2 of the MVT to show that if $f(x)$ is differentiable for all $x \in \mathbf{R}$ and satisfies

$$|f(x) - f(y)| \leq |x - y|^2 \tag{5.3}$$

for all $x, y \in \mathbf{R}$, then $f(x)$ is constant. [*Hint*: Show that (5.3) implies that

$$\lim_{x \to y} \frac{f(x) - f(y)}{x - y} = 0 \tag{5.4}$$

and use the definition of the derivative to interpret the left-hand side of (5.4).]

56. We have seen that

$$f(x) = f_0 e^{rx}$$

satisfies the differential equation

$$\frac{df}{dx} = rf(x)$$

with $f(0) = f_0$. This exercise will show that $f(x)$ is in fact the only solution. Suppose that r is a constant and f is a differentiable function,

$$\frac{df}{dx} = rf(x) \tag{5.5}$$

for all $x \in \mathbf{R}$, and $f(0) = f_0$. The following steps will show that $f(x) = f_0 e^{rx}$, $x \in \mathbf{R}$, is the only solution of (5.5).
(a) Define the function

$$F(x) = f(x)e^{-rx}, \quad x \in \mathbf{R}$$

Use the product rule to show that

$$F'(x) = e^{-rx}[f'(x) - rf(x)]$$

(b) Use (a) and (5.5) to show that $F'(x) = 0$ for all $x \in \mathbf{R}$.
(c) Use Corollary 2 to show that $F(x)$ is a constant and, hence, $F(x) = F(0) = f_0$.
(d) Show that (c) implies that

$$f_0 = f(x)e^{-rx}$$

and therefore,

$$f(x) = f_0 e^{rx}$$

5.2 Monotonicity and Concavity

Fish are indeterminate growers; they increase in body size throughout their life. However, as they become older, they grow proportionately more slowly. Their growth is often described mathematically by the von Bertalanffy equation, which fits a large number of both freshwater and marine fishes. This equation is given by

$$L(x) = L_\infty - (L_\infty - L_0)e^{-Kx}$$

where $L(x)$ denotes the length of the fish at age x, L_0 the length at age 0, and L_∞ the asymptotic maximum attainable length. We assume that $L_\infty > L_0$. K is related to how quickly the fish grows. Figure 5.20 shows examples for two different values of K; L_∞ and L_0 are the same in both cases. We see from the graphs that for larger K, the asymptotic length L_∞ is approached more quickly.

The fact that fish increase their body size throughout their life can be expressed mathematically by the first derivative of the function $L(x)$. Looking at the graph, we see that $L(x)$ is an increasing function of x: The tangent line at any point of the graph has a positive slope, or, equivalently, $L'(x) > 0$. We can compute

$$L'(x) = K(L_\infty - L_0)e^{-Kx}$$

Since $L_\infty > L_0$ (by assumption) and $e^{-Kx} > 0$ (this holds for all x, regardless of K), we see that, indeed, $L'(x) > 0$. The graph of $L'(x)$ is shown in Figure 5.21.

The graph of $L'(x)$ shows that $L'(x)$ is a decreasing function of x: Although fish increase their body size throughout their life, they do so at a rate that decreases with age. Mathematically, this relationship can be expressed with the second derivative of

Section 5.2 Problems

■ 5.2.1 and 5.2.2

In Problems 1–20, determine where each function is increasing, decreasing, concave up, and concave down. With the help of a graphing calculator, sketch the graph of each function and label the intervals where it is increasing, decreasing, concave up, and concave down. Make sure that your graphs and your calculations agree.

1. $y = 3x - x^2, x \in \mathbf{R}$ **2.** $y = x^2 + 5x, x \in \mathbf{R}$

3. $y = x^2 + x - 4, x \in \mathbf{R}$ **4.** $y = x^2 - x + 3, x \in \mathbf{R}$

5. $y = -\frac{2}{3}x^3 + \frac{7}{2}x^2 - 3x + 4, x \in \mathbf{R}$

6. $y = (x-2)^3 + 3, x \in \mathbf{R}$

7. $y = \sqrt{x+1}, x \geq -1$ **8.** $y = (3x-1)^{1/3}, x \in \mathbf{R}$

9. $y = \frac{1}{x}, x \neq 0$ **10.** $y = \frac{-2}{x^2+3}$

11. $(x^2+1)^{1/3}, x \in \mathbf{R}$ **12.** $y = \frac{5}{x-2}, x \neq 2$

13. $y = \frac{1}{(1+x)^2}, x \neq -1$ **14.** $y = \frac{x^2}{x^2+1}, x \geq 0$

15. $y = \sin x, 0 \leq x \leq 2\pi$

16. $y = \cos[\pi(x^2-1)], 2 \leq x \leq 3$

17. $y = e^x, x \in \mathbf{R}$ **18.** $y = \ln x, x > 0$

19. $y = e^{-x^2/2}, x \in \mathbf{R}$ **20.** $y = \frac{1}{1+e^{-x}}, x \in \mathbf{R}$

21. Sketch the graph of
(a) a function that is increasing at an accelerating rate; and
(b) a function that is increasing at a decelerating rate.
(c) Assume that your functions in (a) and (b) are twice differentiable. Explain in each case how you could check the respective properties by using the first and the second derivatives. Which of the functions is concave up, and which is concave down?

22. Show that if $f(x)$ is the linear function $y = mx + b$, then increases in $f(x)$ are proportional to increases in x. That is, if we increase x by Δx, then $f(x)$ increases by the same amount Δy, regardless of the value of x. Compute Δy as a function of Δx.

23. We frequently must solve equations of the form $f(x) = 0$. When f is a continuous function on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, the intermediate-value theorem guarantees that there exists at least one solution of the equation $f(x) = 0$ in $[a, b]$.
(a) Explain in words why there exists exactly one solution in (a, b) if, in addition, f is differentiable in (a, b) and $f'(x)$ is either strictly positive or strictly negative throughout (a, b).
(b) Use the result in (a) to show that

$$x^3 - 4x + 1 = 0$$

has exactly one solution in $[-1, 1]$.

24. First-Derivative Test for Monotonicity Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Show that if $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.

25. Second-Derivative Test for Concavity Suppose that f is twice differentiable on an open interval I. Show that if $f''(x) < 0$, then f is concave down.

26. Suppose the size of a population at time t is $N(t)$, and the growth rate of the population is given by the logistic growth function

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad t \geq 0$$

where r and K are positive constants.
(a) Graph the growth rate $\frac{dN}{dt}$ as a function of N for $r = 3$ and $K = 10$.
(b) The function $f(N) = rN(1 - N/K), N \geq 0$, is differentiable for $N > 0$. Compute $f'(N)$, and determine where the function $f(N)$ is increasing and where it is decreasing.

27. Logistic Growth Suppose that the size of a population at time t is $N(t)$ and the growth rate of the population is given by the logistic growth function

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad t \geq 0$$

where r and K are positive constants. The per capita growth rate is defined by

$$g(N) = \frac{1}{N}\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)$$

(a) Graph $g(N)$ as a function of N for $N \geq 0$ when $r = 3$ and $K = 10$.
(b) The function $g(N) = r(1 - N/K), N \geq 0$, is differentiable for $N > 0$. Compute $g'(N)$, and determine where the function $g(N)$ is increasing and where it is decreasing.

28. Resource-Dependent Growth The growth rate of a plant depends on the amount of resources available. A simple and frequently used model for resource-dependent growth is the Monod model, according to which the growth rate is equal to

$$f(R) = \frac{aR}{k+R}, \quad R \geq 0$$

where R denotes the resource level and a and k are positive constants. When is the growth rate increasing? When is it decreasing?

29. Population Growth Suppose that the growth rate of a population is given by

$$f(N) = N\left(1 - \left(\frac{N}{K}\right)^\theta\right)$$

where N is the size of the population, K is a positive constant denoting the carrying capacity, and θ is a parameter greater than 1. Find $f'(N)$, and determine where the growth rate is increasing and where it is decreasing.

30. Predation Spruce budworms are a major pest that defoliate balsam fir. They are preyed upon by birds. A model for the per capita predation rate is given by

$$f(N) = \frac{aN}{k^2 + N^2}$$

where N denotes the density of spruce budworms and a and k are positive constants. Find $f'(N)$, and determine where the predation rate is increasing and where it is decreasing.

31. Host–Parasitoid Interactions Parasitoids are insects that lay their eggs in, on, or close to other (host) insects. Parasitoid larva then devour the host insect. The likelihood of escaping parasitism may depend on parasitoid density. One model expressing the dependence sets the probability of escaping parasitism equal to

$$f(P) = e^{-aP}$$

where P is the parasitoid density and a is a positive constant. Determine whether the probability of escaping parasitism increases or decreases with parasitoid density.

32. Host–Parasitoid Interactions As an alternative to the model set forth in Problem 31, another model sets the probability of escaping parasitism equal to

$$f(P) = \left(1 + \frac{aP}{k}\right)^{-k}$$

where P is the parasitoid density and a and k are positive constants. Determine whether the probability of escaping parasitism increases or decreases with parasitoid density.

33. Tree Growth Suppose that the height y in feet of a tree as a function of the age x in years of the tree is given by

$$y = 117e^{-10/x}, \quad x > 0$$

(a) Show that the height of the tree increases with age. What is the maximum attainable height?

(b) Where is the graph of height versus age concave up, and where is it concave down?

(c) Use a graphing calculator to sketch the graph of height versus age.

(d) Use a graphing calculator to verify that the rate of growth is greatest at the point where the graph in (c) changes concavity.

34. Reproduction Plants employ two basic reproductive strategies: *polycarpy*, in which reproduction occurs repeatedly during the lifetime of the organism, and *monocarpy*, in which reproduction occurs only once during the lifetime of the organism. (Bamboo, for instance, is a monocarpic plant.) The following quote is taken from Iwasa et al. (1995):

> The optimal strategy is polycarpy (repeated reproduction) if reproductive success increases with the investment at a decreasing rate, [or] monocarpy ("big bang" reproduction) or intermittent reproduction if the reproductive success increases at an increasing rate.

(a) Sketch the graph of reproductive success as a function of reproductive investment for the cases of **(i)** polycarpy and **(ii)** monocarpy.

(b) Given that the second derivative describes whether a curve bends upward or downward, explain the preceding quote in terms of the second derivative of the reproductive success function.

35. Pollinator Visits Assume that the formula (Iwasa et al., 1995)

$$X(F) = cF^{\gamma}$$

where c is a positive constant, expresses the relationship between the number of flowers on a plant, F, and the average number of pollinator visits, $X(F)$. Find the range of values for the parameter γ such that the average number of pollinator visits to a plant increases with the number of flowers F but the rate of increase decreases with F. Explain your answer in terms of appropriate derivatives of the function $X(F)$.

36. Pollinator Visits Assume that the dependence of the average number of pollinator visits to a plant, X, on the number of flowers, F, is given by

$$X(F) = cF^{\gamma}$$

where γ is a positive constant less than 1 and c is a positive constant (Iwasa et al., 1995). How does the average number of pollen grains exported per flower, $E(F)$, change with the number of flowers on the plant, F, if $E(F)$ is proportional to

$$1 - \exp\left[-k\frac{X(F)}{F}\right]$$

where k is a positive constant?

37. Population Size Denote the size of a population by $N(t)$, and assume that $N(t)$ satisfies

$$\frac{dN}{dt} = Ne^{-aN} - N^2$$

where a is a positive constant.

(a) Show that the nontrivial equilibrium N^* satisfies

$$e^{-aN^*} = N^*$$

(b) Assume now that the nontrivial equilibrium N^* is a function of the parameter a. Use implicit differentiation to show that N^* is a decreasing function of a.

38. Population Size Denote the size of a population by $N(t)$, and assume that $N(t)$ satisfies

$$\frac{dN}{dt} = N\left(1 - \frac{N}{K}\right) - N\ln N$$

where K is a positive constant.

(a) Show that if $K > 1$, then there exists a nontrivial equilibrium $N^* > 0$ that satisfies

$$1 - \frac{N^*}{K} = \ln N^*$$

(b) Assume now that the nontrivial equilibrium N^* is a function of the parameter K. Use implicit differentiation to show that N^* is an increasing function of K.

39. Intraspecific Competition *(Adapted from Bellows, 1981)* Suppose that a study plot contains N annual plants, each of which produces S seeds that are sown within the same plot. The number of surviving plants in the next year is given by

$$A(N) = \frac{NS}{1 + (aN)^b} \tag{5.6}$$

for some positive constants a and b. This mathematical model incorporates density-dependent mortality: The greater the number of plants in the plot, the lower is the number of surviving offspring per plant, which is given by $A(N)/N$ and is called the *net reproductive rate*.

(a) Use calculus to show that $A(N)/N$ is a decreasing function of N.

(b) The following quantity, called the *k-value*, can be used to quantify the effects of intraspecific competition (i.e., competition between individuals of the same species):

$$k = \log[\text{initial density}] - \log[\text{final density}]$$

Here, "log" denotes the logarithm to base 10. The initial density is the product of the number of plants (N) and the number of seeds each plant produces (S). The final density is given by (5.6). Use the expression for k and (5.6) to show that

$$\begin{aligned} k &= \log[NS] - \log\left[\frac{NS}{1 + (aN)^b}\right] \\ &= \log\left[1 + (aN)^b\right] \end{aligned}$$

We typically plot k versus $\log N$; the slope of the resulting curve is then used to quantify the effects of competition.

(i) Show that

$$\frac{d\log N}{dN} = \frac{1}{N\ln 10}$$

where ln denotes the natural logarithm.

(ii) Show that

$$\frac{dk}{d\log N} = (\ln 10)N\frac{dk}{dN} = \frac{b}{1+(aN)^{-b}}$$

(iii) Find

$$\lim_{N\to\infty} \frac{dk}{d\log N}$$

(iv) Show that if

$$\frac{dk}{d\log N} < 1$$

then $A(N)$ is increasing, whereas if

$$\frac{dk}{d\log N} > 1$$

then $A(N)$ is decreasing. [*Hint:* Compute $A'(N)$.] Explain in words what the two inequalities mean with respect to varying the initial density of seeds and observing the number of surviving plants the next year. (*Hint:* The first case is called *undercompensation* and the second case is called *overcompensation.*)

(v) The case

$$\frac{dk}{d\log N} = 1$$

is referred to as *exact compensation.* Suppose that you plot k versus $\log N$ and observe that, over a certain range of values of N, the slope of the resulting curve is equal to 1. Explain what this means.

40. *(Adapted from Reiss, 1989)* Suppose that the rate at which body weight W changes with age x is

$$\frac{dW}{dx} \propto W^a \tag{5.7}$$

where a is some species-specific positive constant.

(a) The relative growth rate (percentage weight gained per unit of time) is defined as

$$\frac{1}{W}\frac{dW}{dx}$$

What is the relationship between the relative growth rate and body weight? For which values of a is the relative growth rate increasing, and for which values is it decreasing?

(b) As fish grow larger, their weight increases each day but the relative growth rate decreases. If the rate of growth is described by (5.7), what values of a can you exclude on the basis of your results in (a)? Explain how the increase in percentage weight (relative to the current body weight) differs for juvenile fish and for adult fish.

41. Allometric Growth Allometric equations describe the scaling relationship between two measurements, such as tree height versus tree diameter or skull length versus backbone length. These equations are often of the form

$$Y = bX^a \tag{5.8}$$

where b is some positive constant and a is a constant that can be positive, negative, or zero.

(a) Assume that X and Y are body measurements (and therefore positive) and that their relationship is described by an allometric equation of the form (5.8). For what values of a is Y an increasing function of X, but one such that the ratio Y/X decreases with increasing X? Is Y concave up or concave down in this case?

(b) In vertebrates, we typically find

$$[\text{skull length}] \propto [\text{body length}]^a$$

for some $a \in (0, 1)$. Use your answer in (a) to explain what this means for skull length versus body length in juveniles versus adults; that is, at which developmental stage do vertebrates have larger skulls relative to their body length?

42. pH The pH value of a solution measures the concentration of hydrogen ions, denoted by $[H^+]$, and is defined as

$$\text{pH} = -\log[H^+]$$

Use calculus to decide whether the pH value of a solution increases or decreases as the concentration of H^+ increases.

43. Allometric Growth The differential equation

$$\frac{dy}{dx} = k\frac{y}{x}$$

describes allometric growth, where k is a positive constant. Assume that x and y are both positive variables and that $y = f(x)$ is twice differentiable. Use implicit differentiation to determine for which values of k the function $y = f(x)$ is concave up.

44. Population Size Let $N(t)$ denote the population size at time t, and assume that $N(t)$ is twice differentiable and satisfies the differential equation

$$\frac{dN}{dt} = rN$$

where r is a real number. Differentiate the differential equation with respect to t, and state whether $N(t)$ is concave up or down.

5.3 Extrema, Inflection Points, and Graphing

5.3.1 Extrema

If f is a continuous function on the closed interval $[a, b]$, then f has a global maximum and a global minimum in $[a, b]$. This is the content of the extreme-value theorem, which is an existence result: It tells us only that global extrema exist under certain conditions, but it does not tell us how to find them.

Our strategy for finding global extrema in the case where f is a continuous function defined on a closed interval will be, first, to identify all local extrema of the function and, then, to select the global extrema from the set of local extrema. If f is a continuous function defined on an open interval or half-open interval, the existence of global extrema is no longer guaranteed, and we must compare the local extrema with the behavior of the function near the open boundaries of the domain. (See Example 5 in Section 5.1.) In particular, if $f(x)$ is defined on $\mathbf{R}$, we need to

We find that $f(x)$ is concave up for $x < -1$ and $x > 1$ and is concave down for $-1 < x < 1$. There are two inflection points, one at $x = -1$, namely, $(-1, e^{-1/2})$, and the other at $x = 1$, namely, $(1, e^{-1/2})$. There are no other inflection points, since $f''(x)$ is defined for all $x \in \mathbf{R}$.

STEP 4. We have

$$\lim_{x\to-\infty} f(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} f(x) = 0$$

This shows that $y = 0$ is a horizontal asymptote.

The graph of $f(x)$ is shown in Figure 5.49. ■

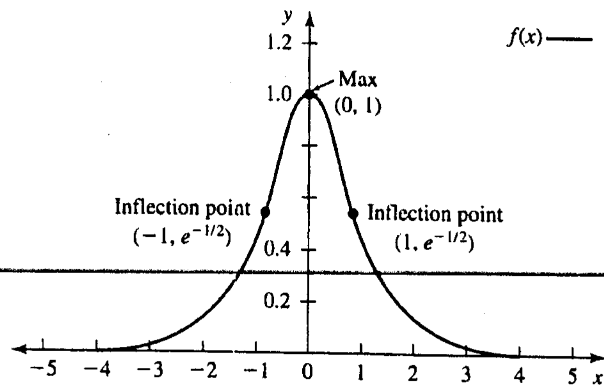

Figure 5.49 The graph of $f(x) = e^{-x^2/2}$.

Section 5.3 Problems

■ 5.3.1

Find the local maxima and minima of each of the functions in Problems 1–16. Determine whether each function has absolute maxima and minima and find their coordinates. For each function, find the intervals on which it is increasing and the intervals on which it is decreasing.

1. $y = (2-x)^2, -2 \le x \le 3$
2. $y = \sqrt{x-1}, 1 \le x \le 2$
3. $y = \ln(2x-1), 1 \le x \le 2$
4. $y = \ln \frac{x}{x+1}, x > 0$
5. $y = xe^{-x}, 0 \le x \le 1$
6. $y = |16 - x^2|, -5 \le x \le 8$
7. $y = (x-1)^3 + 1, x \in \mathbf{R}$
8. $y = x^3 - 3x + 1, x \in \mathbf{R}$
9. $y = \cos(\pi x^2), -1 \le x \le 1$
10. $y = \sin[2\pi(x-3)], 2 \le x \le 3$
11. $y = e^{-|x|}, x \in \mathbf{R}$
12. $y = e^{-x^2/4}, x \in \mathbf{R}$
13. $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 2, x \in \mathbf{R}$
14. $y = x^2(1-x), x \in \mathbf{R}$
15. $y = (x-1)^{1/3}, x \in \mathbf{R}$
16. $y = \sqrt{1+x^2}, x \in \mathbf{R}$
17. [This problem illustrates the fact that $f'(c) = 0$ is not a sufficient condition for the existence of a local extremum of a differentiable function.] Show that the function $f(x) = x^3$ has a horizontal tangent at $x = 0$; that is, show that $f'(0) = 0$, but $f'(x)$ does not change sign at $x = 0$ and, hence, $f(x)$ does not have a local extremum at $x = 0$.
18. Suppose that $f(x)$ is twice differentiable on $\mathbf{R}$, with $f(x) > 0$ for $x \in \mathbf{R}$. Show that if $f(x)$ has a local maximum at $x = c$, then $g(x) = \ln f(x)$ also has a local maximum at $x = c$.

■ 5.3.2

In Problems 19–24, determine all inflection points.

19. $f(x) = x^3 - 2, x \in \mathbf{R}$
20. $f(x) = (x-3)^5, 0 \in \mathbf{R}$
21. $f(x) = e^{-x^2}, x \ge 0$
22. $f(x) = xe^{-x}, x \ge 0$
23. $f(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$
24. $f(x) = \ln x + \frac{1}{x}, x > 0$
25. [This problem illustrates the fact that $f''(c) = 0$ is not a sufficient condition for an inflection point of a twice-differentiable function.] Show that the function $f(x) = x^4$ has $f''(0) = 0$ but that $f''(x)$ does not change sign at $x = 0$ and, hence, $f(x)$ does not have an inflection point at $x = 0$.
26. **Logistic Equation** Suppose that the size of a population at time t is denoted by $N(t)$ and satisfies

$$N(t) = \frac{100}{1 + 3e^{-2t}}$$

for $t \ge 0$.

(a) Show that $N(0) = 25$.

(b) Show that $N(t)$ is strictly increasing.

(c) Show that

$$\lim_{t\to\infty} N(t) = 100$$

(d) Show that $N(t)$ has an inflection point when $N(t) = 50$—that is, when the size of the population is at half its limiting value.

(e) Use your results in (a)–(d) to sketch the graph of $N(t)$.

■ 5.3.3

Find the local maxima and minima of the functions in Problems 27–34. Determine whether the functions have absolute maxima and minima, and, if so, find their coordinates. Find inflection points. Find the intervals on which the function is increasing, on which it is decreasing, on which it is concave up, and on which it is concave down. Sketch the graph of each function.

27. $y = \frac{2}{3}x^3 - 2x^2 - 6x + 2$ for $-2 \le x \le 5$

28. $y = x^4 - 2x^2, x \in \mathbf{R}$

29. $y = |x^2 - 9|, -4 \le x \le 5$

30. $y = \sqrt{|x|}, x \in \mathbf{R}$

31. $y = x + \cos x, x \in \mathbf{R}$

32. $y = \tan x - x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

33. $y = \frac{x^2 - 1}{x^2 + 1}, x \in \mathbf{R}$

34. $y = \ln(x^2 + 1), x \in \mathbf{R}$

35. Let

$$f(x) = \frac{x}{x-1}, \quad x \neq 1$$

(a) Show that

$$\lim_{x \to -\infty} f(x) = 1$$

and

$$\lim_{x \to +\infty} f(x) = 1$$

That is, show that $y = 1$ is a horizontal asymptote of the curve $y = \frac{x}{x-1}$.

(b) Show that

$$\lim_{x \to 1^-} f(x) = -\infty$$

and

$$\lim_{x \to 1^+} f(x) = +\infty$$

That is, show that $x = 1$ is a vertical asymptote of the curve $y = \frac{x}{x-1}$.

(c) Determine where $f(x)$ is increasing and where it is decreasing. Does $f(x)$ have local extrema?

(d) Determine where $f(x)$ is concave up and where it is concave down. Does $f(x)$ have inflection points?

(e) Sketch the graph of $f(x)$ together with its asymptotes.

36. Let

$$f(x) = -\frac{2}{x^2 - 1}, \quad x \neq -1, 1$$

(a) Show that

$$\lim_{x \to +\infty} f(x) = 0$$

and

$$\lim_{x \to -\infty} f(x) = 0$$

That is, show that $y = 0$ is a horizontal asymptote of $f(x)$.

(b) Show that

$$\lim_{x \to -1^-} f(x) = -\infty$$

and

$$\lim_{x \to -1^+} f(x) = +\infty$$

and that

$$\lim_{x \to 1^-} f(x) = +\infty$$

and

$$\lim_{x \to 1^+} f(x) = -\infty$$

That is, show that $x = -1$ and $x = 1$ are vertical asymptotes of $f(x)$.

(c) Determine where $f(x)$ is increasing and where it is decreasing. Does $f(x)$ have local extrema?

(d) Determine where $f(x)$ is concave up and where it is concave down. Does $f(x)$ have inflection points?

(e) Sketch the graph of $f(x)$ together with its asymptotes.

37. Let

$$f(x) = \frac{2x^2 - 5}{x + 2}, \quad x \neq -2$$

(a) Show that $x = -2$ is a vertical asymptote.

(b) Determine where $f(x)$ is increasing and where it is decreasing. Does $f(x)$ have local extrema?

(c) Determine where $f(x)$ is concave up and where it is concave down. Does $f(x)$ have inflection points?

(d) Since the degree of the numerator is one higher than the degree of the denominator, $f(x)$ has an oblique asymptote. Find it.

(e) Sketch the graph of $f(x)$ together with its asymptotes.

38. Let

$$f(x) = \frac{\sin x}{x}, \quad x \neq 0$$

(a) Show that $y = 0$ is a horizontal asymptote.

(b) Since $f(x)$ is not defined at $x = 0$, does this mean that $f(x)$ has a vertical asymptote at $x = 0$? Find $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$.

(c) Use a graphing calculator to sketch the graph of $f(x)$.

39. Let

$$f(x) = \frac{x^2}{1 + x^2}, \ x \in \mathbf{R}$$

(a) Determine where $f(x)$ is increasing and where it is decreasing.

(b) Where is the function concave up and where is it concave down? Find all inflection points of $f(x)$.

(c) Find $\lim_{x \to \pm\infty} f(x)$ and decide whether $f(x)$ has a horizontal asymptote.

(d) Sketch the graph of $f(x)$ together with its asymptotes and inflection points (if they exist).

40. Let

$$f(x) = \frac{x^k}{1 + x^k}, \ x \ge 0$$

where k is a positive integer greater than 1.

(a) Determine where $f(x)$ is increasing and where it is decreasing.

(b) Where is the function concave up and where is it concave down? Find all inflection points of $f(x)$.

(c) Find $\lim_{x \to \infty} f(x)$ and decide whether $f(x)$ has a horizontal asymptote.

(d) Sketch the graph of $f(x)$ together with its asymptotes and inflection points (if they exist).

41. Let

$$f(x) = \frac{x}{a + x}, \ x \ge 0$$

where a is a positive constant.

(a) Determine where $f(x)$ is increasing and where it is decreasing.

(b) Where is the function concave up and where is it concave down? Find all inflection points of $f(x)$.

(c) Find $\lim_{x\to\infty} f(x)$ and decide whether $f(x)$ has a horizontal asymptote.

(d) Sketch the graph of $f(x)$ together with its asymptotes and inflection points (if they exist).

42. Let

$$f(x) = \frac{2}{1+e^{-x}}, \quad x \in \mathbf{R}$$

(a) Determine where $f(x)$ is increasing and where it is decreasing.

(b) Where is the function concave up and where is it concave down? Find all inflection points of $f(x)$.

(c) Find $\lim_{x\to\infty} f(x)$ and decide whether $f(x)$ has a horizontal asymptote.

(d) Find $\lim_{x\to-\infty} f(x)$ and decide whether $f(x)$ has a horizontal asymptote.

(e) Sketch the graph of $f(x)$ together with its asymptotes and inflection points (if they exist).

43. **Population Growth** Suppose that the growth rate of a population is given by

$$f(N) = N\left(1 - \left(\frac{N}{K}\right)^{\theta}\right) \quad N \geq 0$$

where N is the size of the population, K is a positive constant denoting the carrying capacity, and θ is a parameter greater than 1. Find the population size for which the growth rate is maximal.

44. **Predation Rate** Spruce budworms are a major pest that defoliate balsam fir. They are preyed upon by birds. A model for the per capita predation rate is given by

$$f(N) = \frac{aN}{k^2 + N^2}$$

where N denotes the density of spruce budworm and a and k are positive constants. For which density of spruce budworms is the per capita predation rate maximal?

5.4 Optimization

There are many situations in which we wish to maximize or minimize certain quantities. For instance, in a chemical reaction, you might wish to know under which conditions the reaction rate is maximized. In an agricultural setting, you might be interested in finding the amount of fertilizer that would maximize the yield of some crops. In a medical setting, you might wish to optimize the dosage of a drug for maximum benefit. Optimization problems also arise in the study of the evolution of life histories and involve questions such as when an organism should begin reproduction in order to maximize the number of surviving offspring. In each case, we are interested in finding global extrema.

EXAMPLE 1 **Chemical Reaction** Consider the chemical reaction

$$A + B \rightarrow AB$$

In Example 5 of Subsection 1.2.2, we found that the reaction rate is given by the function

$$R(x) = k(a-x)(b-x), \quad 0 \leq x \leq \min(a,b)$$

where x is the concentration of the product AB and $\min(a,b)$ denotes the minimum of the two values of a and b. The constants a and b are the concentrations of the reactants A and B at the beginning of the reaction. To be concrete, we choose $k = 2$, $a = 2$, and $b = 5$. Then

$$R(x) = 2(2-x)(5-x) \quad \text{for } 0 \leq x \leq 2$$

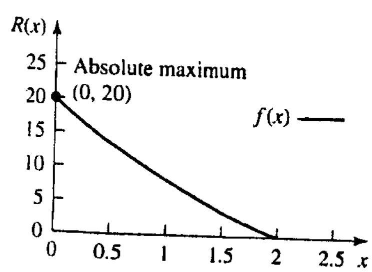

Figure 5.50 The chemical reaction rate $R(x)$ in Example 1. The graph of $R(x) = 2(2-x)(5-x)$, $0 \leq x \leq 2$, has an absolute maximum at $(0, 20)$.

(See Figure 5.50.)

We are interested in finding the concentration x that maximizes the reaction rate; this is the absolute maximum of $R(x)$. Since $R(x)$ is differentiable on $(0, 2)$, we can find all local extrema on $(0, 2)$ by investigating the first derivative. To compute the first derivative of $R(x)$, we multiply $R(x)$ out:

$$R(x) = 20 - 14x + 2x^2 \quad \text{for } 0 \leq x \leq 2$$

Differentiating with respect to x yields

$$R'(x) = -14 + 4x \quad \text{for } 0 < x < 2$$

we conclude that there is a local maximum at $\hat{x} = k$. To see whether it is a global maximum, we compare $w(k)$ with $w(0)$ and $\lim_{x\to\infty} w(x)$. We have

$$w(x) = \frac{R}{x} f(x) = \frac{R}{x}\frac{x^2}{k^2+x^2} = R\frac{x}{k^2+x^2}$$

so

$$w(0) = 0 \qquad w(k) = \frac{R}{2k} \qquad \lim_{x\to\infty} w(x) = 0$$

Hence, $\hat{x} = k$ is where the absolute maximum occurs; for our choice of $f(x) = \frac{x^2}{k^2+x^2}$, the optimal clutch size N_{opt} satisfies $N_{\text{opt}} = R/k$. [Other choices of $f(x)$ would give a different result.]

There is a geometric way of finding $\hat{x}$. Since

$$f'(\hat{x}) = \frac{f(\hat{x})}{\hat{x}}$$

it follows that the tangent line at $(\hat{x}, f(\hat{x}))$ has slope $\frac{f(\hat{x})}{\hat{x}}$. This line can be obtained by drawing a straight line through the origin that just touches the graph of $y = f(x)$, as illustrated in Figure 5.55. ■

Section 5.4 Problems

1. Find the smallest perimeter possible for a rectangle whose area is 25 in.^{2}.

2. Show that, among all rectangles with a given perimeter, the square has the largest area.

3. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 3 - x^2$, as shown in Figure 5.56. What is the largest area the rectangle can have?

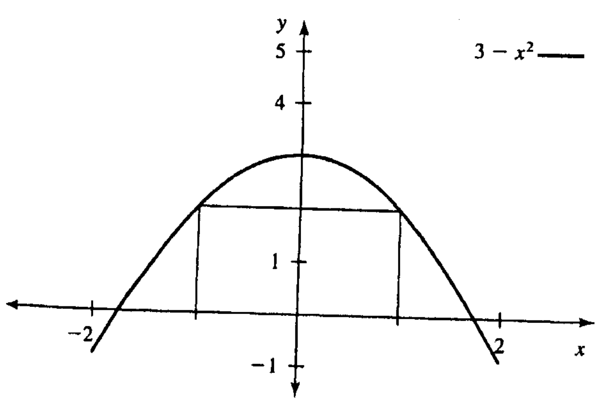

Figure 5.56 The graph of $y = 3 - x^2$ together with the inscribed rectangle in Problem 3.

4. A rectangular study area is to be enclosed by a fence and divided into two equal parts, with the fence running along the division parallel to one of the sides. If the total area is 384 ft^2, find the dimensions of the study area that will minimize the total length of the fence. How much fencing will be required?

5. A rectangular field is bounded on one side by a river and on the other three sides by a fence. Find the dimensions of the field that will maximize the enclosed area if the fence has a total length of 320 ft.

6. Find the largest possible area of a right triangle whose hypotenuse is 4 cm long.

7. Suppose that a and b are the side lengths in a right triangle whose hypotenuse is 5 cm long. What is the largest perimeter possible?

8. Suppose that a and b are the side lengths in a right triangle whose hypotenuse is 10 cm long. Show that the area of the triangle is largest when $a = b$.

9. A rectangle has its base on the x-axis, its lower left corner at (0, 0), and its upper right corner on the curve $y = 1/x$. What is the smallest perimeter the rectangle can have?

10. A rectangle has its base on the x-axis and its upper left and right corners on the curve $y = \sqrt{4 - x^2}$, as shown in Figure 5.57. The left and the right corners are equidistant from the vertical axis. What is the largest area the rectangle can have?

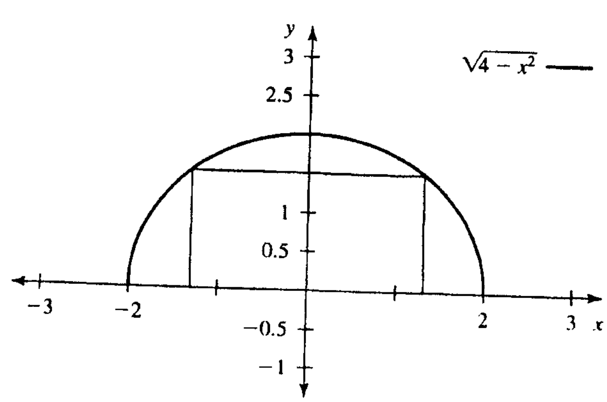

Figure 5.57 The graph of $y = (4 - x^2)^{1/2}$ together with the inscribed rectangle in Problem 10.

11. Denote by (x, y) a point on the straight line $y = 4 - 3x$. (See Figure 5.58.)

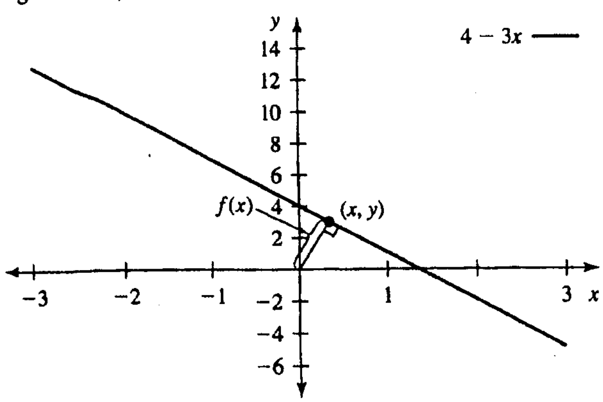

Figure 5.58 The graph of $y = 4 - 3x$ in Problem 11.

(a) Show that the distance from (x, y) to the origin is given by

$$f(x) = \sqrt{x^2 + (4 - 3x)^2}$$

(b) Give the coordinates of the point on the line $y = 4 - 3x$ that is closest to the origin. (*Hint*: Find x so that the distance you computed in (a) is minimized.)

(c) Show that the *square* of the distance between the point (x, y) on the line and the origin is given by

$$g(x) = [f(x)]^2 = x^2 + (4 - 3x)^2$$

and find the minimum of $g(x)$. Show that this minimum agrees with your answer in (b).

12. How close does the line $y = 1 + 2x$ come to the origin?

13. How close does the curve $y = 1/x$ come to the origin? (*Hint*: Find the point on the curve that minimizes the *square* of the distance between the origin and the point on the curve. If you use the square of the distance instead of the distance, you avoid dealing with square roots.)

14. How close does the circle with radius $\sqrt{2}$ and center $(2, 2)$ come to the origin.

15. Show that if $f(x)$ is a positive twice-differentiable function that has a local minimum at $x = c$, then $g(x) = [f(x)]^2$ has a local minimum at $x = c$ as well.

16. Show that if $f(x)$ is a differentiable function with $f(x) < 0$ for all $x \in \mathbf{R}$ and with a local maximum at $x = c$, then $g(x) = [f(x)]^2$ has a local minimum at $x = c$.

17. Find the dimensions of a right circular cylindrical can (with bottom and top closed) that has a volume of 1 liter and that minimizes the amount of material used. (*Note*: One liter corresponds to 1000 cm^3.)

18. Find the dimensions of a right circular cylinder that is open on the top, is closed on the bottom, holds 1 liter, and uses the least amount of material.

19. A circular sector with radius r and angle θ has area A. Find r and θ so that the perimeter is smallest when (a) $A = 2$ and (b) $A = 10$. (*Note*: $A = \frac{1}{2}r^2\theta$, and the length of the arc $s = r\theta$, when θ is measured in radians; see Figure 5.59.)

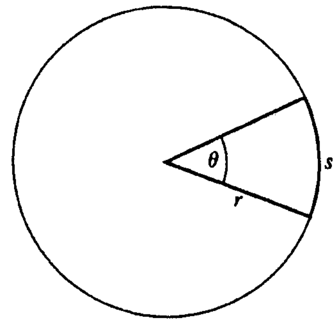

Figure 5.59 The circular sector in Problems 19 and 20.

20. A circular sector with radius r and angle θ has area A. Find r and θ so that the perimeter is smallest for a given area A. (*Note*: $A = \frac{1}{2}r^2\theta$, and the length of the arc $s = r\theta$, when θ is measured in radians; see Figure 5.59.)

21. Repeat Example 4 under the assumption that the top of the can is made out of aluminum that is three times as thick as the aluminum used for the wall and the bottom.

22. Find two positive numbers a and b such that $a + b = 20$ and ab is a maximum.

23. Find two numbers a and b such that $a - b = 4$ and ab is a minimum.

24. Classical Model of Viability Selection Consider a population of diploid organisms (i.e., each individual carries two copies of each chromosome). Genes reside on chromosomes, and we call the location of a gene on a chromosome a **locus**. Different versions of the same gene are called **alleles**. Let us examine the case of one locus with two possible alleles, A_1 and A_2. Since the individuals are diploid, the following types, called **genotypes**, may occur: A_1A_1, A_1A_2, and A_2A_2 (where A_1A_2 and A_2A_1 are considered to be equivalent). If two parents mate and produce an offspring, the offspring receives one gene from each parent. If mating is random, then we can imagine all genes being put into one big gene pool from which we choose two genes at random. If we assume that the frequency of A_1 in the population is p and the frequency of A_2 is $q = 1 - p$, then the combination A_1A_1 is picked with probability p^2, the combination A_1A_2 with probability $2pq$ (the factor 2 appears because A_1 can come from either the father or the mother), and the combination A_2A_2 with probability q^2.

We assume that the survival chances of offspring depend on their genotypes. We define the quantities w_{11}, w_{12}, and w_{22} to describe the differential survival chances of the types A_1A_1, A_1A_2, and A_2A_2, respectively. The ratio $A_1A_1{:}A_1A_2{:}A_2A_2$ among adults is given by

$$p^2w_{11}:2pqw_{12}:q^2w_{22}$$

The average fitness of this population is defined as

$$\overline{w} = p^2w_{11} + 2pqw_{12} + q^2w_{22}$$

We will investigate the preceding function. Since $q = 1 - p$, $\overline{w}$ is a function of p only; specifically,

$$\overline{w}(p) = p^2w_{11} + 2p(1 - p)w_{12} + (1 - p)^2w_{22}$$

for $0 \le p \le 1$. We consider the following three cases:

(i) Directional selection: $w_{11} > w_{12} > w_{22}$

(ii) Overdominance: $w_{12} > w_{11}, w_{22}$

(iii) Underdominance: $w_{12} < w_{11}, w_{22}$

(a) Show that

$$\overline{w}(p) = p^2(w_{11} - 2w_{12} + w_{22}) + 2p(w_{12} - w_{22}) + w_{22}$$

and graph $\overline{w}(p)$ for each of the three cases, where we choose the parameters as follows:
(i) $w_{11} = 1, w_{12} = 0.7, w_{22} = 0.3$
(ii) $w_{11} = 0.7, w_{12} = 1, w_{22} = 0.3$
(iii) $w_{11} = 1, w_{12} = 0.3, w_{22} = 0.7$
(b) Show that

$$\frac{d\overline{w}}{dp} = 2p(w_{11} - 2w_{12} + w_{22}) + 2(w_{12} - w_{22})$$

(c) Find the global maximum of $\overline{w}(p)$ in each of the three cases considered in (a). (Note that the global maximum may occur at the boundary of the domain of $\overline{w}$.)
(d) We can show that under a certain mating scheme the gene frequencies change until $\overline{w}$ reaches its global maximum. Assume that this is the case, and state what the equilibrium frequency will be for each of the three cases considered in (a).

25. Continuation of Problem 94 from Section 4.3 We discussed the properties of hatching offspring per unit time, $w(t)$, in the species *Eleutherodactylus coqui*. The function $w(t)$ was given by

$$w(t) = \frac{f(t)}{C + t}$$

where $f(t)$ is the proportion of offspring that survive if t is the time spent brooding and where C is the cost associated with the time spent searching for other mates.

We assume now that $f(t)$, $t \geq 0$, is twice differentiable and concave down with $f(0) = 0$ and $0 \leq f \leq 1$. The optimal brooding time is defined as the time that maximizes $w(t)$.
(a) Show that the optimal brooding time can be obtained by finding the point on the curve $f(t)$ where the line through $(-C, 0)$ is tangential to the curve $f(t)$.
(b) Use the procedure in (a) to find the optimal brooding time for $f(t) = \frac{t}{1+t}$ and $C = 2$. Determine the equation of the line through $(-2, 0)$ that is tangential to the curve $f(t) = \frac{t}{1+t}$, and graph both $f(t)$ and the tangent together.

26. Optimal Age of Reproduction *(from Roff, 1992)* Semelparous organisms breed only once during their lifetime. Examples of this type of reproduction can be found in Pacific salmon and bamboo. The per capita rate of increase, r, can be thought of as a measure of reproductive fitness. The greater the value of r, the more offspring an individual produces. The intrinsic rate of increase is typically a function of age x. Models for age-structured populations of semelparous organisms predict that the intrinsic rate of increase as a function of x is given by

$$r(x) = \frac{\ln[l(x)m(x)]}{x}$$

where $l(x)$ is the probability of surviving to age x and $m(x)$ is the number of female offspring at age x. The optimal age of reproduction is the age x that maximizes $r(x)$.
(a) Find the optimal age of reproduction for

$$l(x) = e^{-ax}$$

and

$$m(x) = bx^c$$

where a, b, and c are positive constants.
(b) Use a graphing calculator to sketch the graph of $r(x)$ when $a = 0.1$, $b = 4$, and $c = 0.9$.

27. Optimal Age at First Reproduction *(from Lloyd, 1987)* Iteroparous organisms breed more than once during their lifetime. Consider a model in which the intrinsic rate of increase, r, depends on the age of first reproduction, denoted by x, and satisfies the equation

$$\frac{e^{-x(r(x)+L)}(1 - e^{-kx})^3 c}{1 - e^{-(r(x)+L)}} = 1 \qquad (5.13)$$

where k, L, and c are positive constants describing the life history of the organism. The optimal age of first reproduction is the age x for which $r(x)$ is maximized. Since we cannot separate $r(x)$ in the preceding equation, we must use implicit differentiation to find a candidate for the optimal age of reproduction.
(a) Find an equation for $\frac{dr}{dx}$. [*Hint:* Take logarithms of both sides of (5.13) before differentiating with respect to x.]
(b) Set $\frac{dr}{dx} = 0$ and show that this gives

$$r(x) = \frac{3ke^{-kx}}{1 - e^{-kx}} - L$$

[To find the candidate for the optimal age x, you would need to substitute for $r(x)$ in (5.13) and solve the equation numerically. Then you would still need to check that this solution actually gives you the absolute maximum. It can, in fact, be done.]

5.5 L'Hospital's Rule

Guillaume François l'Hospital was born in France in 1661. He became interested in calculus around 1690, when articles on the new calculus by Leibniz and the Bernoulli brothers began to appear. Johann Bernoulli was in Paris in 1691, and l'Hospital asked Bernoulli to teach him some calculus. Bernoulli left Paris a year later, but continued to provide l'Hospital with new material on calculus. Bernoulli received a monthly salary for his service and agreed that he would not give anyone else access to the material. Once l'Hospital thought he understood the material well enough, he decided to write a book on the subject, which was published under his name and met with great success. Bernoulli was not particularly happy about this development, as his contributions were hardly acknowledged in the book; l'Hospital perhaps felt that because he had paid for the course material, he had a right to publish it.

The limit is now of the form $\infty \cdot 0$ (since $\ln \tan \frac{\pi}{4} = \ln 1 = 0$). We evaluate the limit by writing it in the form $\frac{0}{0}$ and then applying l'Hospital's rule:

$$\lim_{x \to (\pi/4)^-} (\tan(2x) \cdot \ln \tan x) = \lim_{x \to (\pi/4)^-} \frac{\ln \tan x}{\frac{1}{\tan(2x)}} = \lim_{x \to (\pi/4)^-} \frac{\ln \tan x}{\cot(2x)}$$

Since

$$\frac{d}{dx} \ln \tan x = \frac{\sec^2 x}{\tan x} = \frac{\cos x}{\cos^2 x \sin x} = \frac{1}{\sin x \cos x}$$

and

$$\frac{d}{dx} \cot(2x) = -(\csc^2(2x)) \cdot 2 = \frac{-2}{\sin^2(2x)}$$

it follows that

$$\lim_{x \to (\pi/4)^-} \frac{\ln \tan x}{\cot(2x)} = \lim_{x \to (\pi/4)^-} \frac{\frac{1}{\sin x \cos x}}{\frac{-2}{\sin^2(2x)}} = \lim_{x \to (\pi/4)^-} \frac{\sin^2(2x)}{-2 \sin x \cos x}$$

$$= \frac{1}{(-2)(\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{2})} = -1$$

Therefore,

$$\lim_{x \to (\pi/4)^-} (\tan x)^{\tan(2x)} = \exp\left[\lim_{x \to (\pi/4)^-} (\tan(2x) \ln \tan x)\right]$$

$$= \exp[-1] = e^{-1}$$

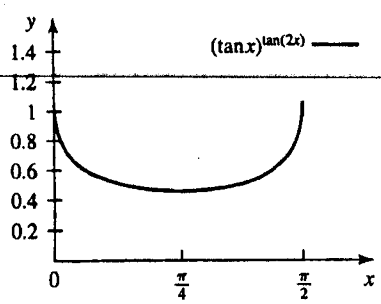

Figure 5.63 The graph of $y = (\tan x)^{\tan(2x)}$.

The graph of $f(x) = (\tan x)^{\tan(2x)}$ is shown in Figure 5.63. ■

Section 5.5 Problems

Use l'Hospital's rule to find the limits in Problems 1–50.

1. $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$
2. $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$
3. $\lim_{x \to -2} \frac{3x^2 + 5x - 2}{x + 2}$
4. $\lim_{x \to -3} \frac{x + 3}{x^2 + 2x - 3}$
5. $\lim_{x \to 0} \frac{\sqrt{2x + 4} - 2}{x}$
6. $\lim_{x \to 0} \frac{3 - \sqrt{2x + 9}}{2x}$
7. $\lim_{x \to 0} \frac{\sin x}{x \cos x}$
8. $\lim_{x \to 0} \frac{x \sin x}{1 - \cos x}$
9. $\lim_{x \to 0} \frac{1 - \cos x}{x \tan x}$
10. $\lim_{x \to \pi/2} \frac{\sin(\frac{\pi}{2} - x)}{\cos x}$
11. $\lim_{x \to 0^+} \frac{\sqrt{x}}{\ln(x + 1)}$
12. $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$
13. $\lim_{x \to \infty} \frac{\ln(\ln x)}{x}$
14. $\lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x}$
15. $\lim_{x \to 0} \frac{2^x - 1}{3^x - 1}$
16. $\lim_{x \to 0} \frac{5^x - 1}{7^x - 1}$
17. $\lim_{x \to 0} \frac{3^{-x} - 1}{2^x - 1}$
18. $\lim_{x \to 0} \frac{2^{-x} - 1}{5^x - 1}$
19. $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$
20. $\lim_{x \to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$
21. $\lim_{x \to \infty} \frac{(\ln x)^2}{x^2}$
22. $\lim_{x \to \infty} \frac{x^7}{e^x}$
23. $\lim_{x \to (\pi/2)^-} \frac{\tan x}{\sec^2 x}$
24. $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$
25. $\lim_{x \to \infty} x e^{-x}$
26. $\lim_{x \to \infty} x^2 e^{-x}$
27. $\lim_{x \to \infty} x^5 e^{-x}$
28. $\lim_{x \to \infty} x^n e^{-x}, n \in \mathbf{N}$
29. $\lim_{x \to 0^+} \sqrt{x} \ln x$
30. $\lim_{x \to 0^+} x^2 \ln x$
31. $\lim_{x \to 0^+} x^5 \ln x$
32. $\lim_{x \to 0^+} x^n \ln x, n \in \mathbf{N}$
33. $\lim_{x \to (\pi/2)^-} \left(\frac{\pi}{2} - x\right) \sec x$
34. $\lim_{x \to 1^-} (1 - x) \tan\left(\frac{\pi}{2} x\right)$
35. $\lim_{x \to \infty} \sqrt{x} \sin \frac{1}{x}$
36. $\lim_{x \to \infty} x^2 \sin \frac{1}{x^2}$
37. $\lim_{x \to 0^+} (\cot x - \csc x)$
38. $\lim_{x \to \infty} (x - \sqrt{x^2 - 1})$
39. $\lim_{x \to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$
40. $\lim_{x \to 0^+} \left(\frac{1}{\sin^2 x} - \frac{1}{x}\right)$

41. $\lim_{x\to 0^+} x^{2x}$

42. $\lim_{x\to 0^+} x^{\sin x}$

43. $\lim_{x\to\infty} x^{1/x}$

44. $\lim_{x\to\infty} (1+e^x)^{1/x}$

45. $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x$

46. $\lim_{x\to\infty} \left(1+\frac{5}{x}\right)^x$

47. $\lim_{x\to\infty} \left(1-\frac{2}{x}\right)^x$

48. $\lim_{x\to\infty} \left(1+\frac{3}{x^2}\right)^x$

49. $\lim_{x\to\infty} \left(\frac{x}{1+x}\right)^x$

50. $\lim_{x\to 0^+} (\cos(2x))^{3/x}$

Find the limits in Problems 51–60. Be sure to check whether you can apply l'Hospital's rule before you evaluate the limit.

51. $\lim_{x\to 0} xe^x$

52. $\lim_{x\to 0^+} \frac{e^x}{x}$

53. $\lim_{x\to(\pi/2)^-} (\tan x + \sec x)$

54. $\lim_{x\to(\pi/2)^-} \frac{\tan x}{1+\sec x}$

55. $\lim_{x\to 1} \frac{x^2-1}{x+1}$

56. $\lim_{x\to 0} \frac{1-\cos x}{\sec x}$

57. $\lim_{x\to -\infty} xe^{x}$

58. $\lim_{x\to 0^+} \left(\frac{1}{x}-\frac{1}{\sqrt{x}}\right)$

59. $\lim_{x\to 0^+} x^{3x}$

60. $\lim_{x\to\infty} \left(\frac{x+1}{x+2}\right)^x$

61. Use l'Hospital's rule to find

$$\lim_{x\to 0} \frac{a^x-1}{b^x-1}$$

where $a, b > 0$.

62. Use l'Hospital's rule to find

$$\lim_{x\to\infty} \left(1+\frac{c}{x}\right)^x$$

where c is a constant.

63. For $p > 0$, determine the values of p for which the following limit is either 1 or ∞ or a constant that is neither 1 nor ∞:

$$\lim_{x\to\infty} \left(1+\frac{c}{x^p}\right)^x$$

64. Show that

$$\lim_{x\to\infty} x^p e^{-x} = 0$$

for any positive number p. Graph $f(x) = x^p e^{-x}$, $x > 0$, for $p = 1/2$, 1, and 2. Since $f(x) = x^p e^{-x} = x^p/e^x$, the limiting behavior ($\lim_{x\to\infty} \frac{x^p}{e^x} = 0$) shows that the exponential function grows faster than any power of x as $x \to \infty$.

65. Show that

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$$

for any number $p > 0$. This shows that the logarithmic function grows more slowly than any positive power of x as $x \to \infty$.

66. When l'Hospital introduced indeterminate limits in his textbook, his *first* example was

$$\lim_{x\to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

where a is a positive constant. (This example was communicated to him by Bernoulli.) Show that this limit is equal to $(16/9)a$.

67. The height y in feet of a tree as a function of the tree's age x in years is given by

$$y = 121e^{-17/x} \quad \text{for } x > 0$$

(a) Determine (1) the rate of growth when $x \to 0^+$ and (2) the limit of the height as $x \to \infty$.

(b) Find the age at which the growth rate is maximal.

(c) Show that the height of the tree is an increasing function of age. At what age is the height increasing at an accelerating rate and at what age at a decelerating rate?

(d) Sketch the graph of both the height and the rate of growth of the tree as functions of age.

■ 5.6 Difference Equations: Stability (Optional)

In Chapter 2, we introduced difference equations and saw that first-order difference equations can be described by recursions of the form

$$x_{t+1} = f(x_t), \quad t = 0, 1, 2, \ldots \tag{5.14}$$

where $f(x)$ is a function. There, we were able to analyze difference equations only numerically (except for equations describing exponential growth, which we were able to solve). We saw that fixed points (or equilibria) played a special role. A fixed point x^* of (5.14) satisfies the equation

$$x^* = f(x^*) \tag{5.15}$$

and has the property that if $x_0 = x^*$, then $x_t = x^*$ for $t = 1, 2, 3, \ldots$. We also saw in a number of applications that, under certain conditions, x_t converged to the fixed point as $t \to \infty$ even if $x_0 \neq x^*$. However, back then, we were not able to predict when this behavior would occur.

In this section, we will return to fixed points and use calculus to come up with a condition that allows us to check whether convergence to a fixed point occurs. We start with the simplest example: exponential growth.

Using the product rule and the chain rule, we find that

$$f'(N) = \exp\left[R\left(1-\frac{N}{K}\right)\right] + N\exp\left[R\left(1-\frac{N}{K}\right)\right]\left(-\frac{R}{K}\right)$$
$$= \exp\left[R\left(1-\frac{N}{K}\right)\right]\left(1-\frac{NR}{K}\right)$$

Now,

$$f'(0) = e^R > 1$$

for $R > 0$, so $N^* = 0$ is unstable. Since

$$f'(K) = 1 - R$$

and $|f'(K)| = |1 - R| < 1$ if $-1 < 1 - R < 1$ or $0 < R < 2$, we conclude that $N^* = K$ is locally stable if $0 < R < 2$. We can say a bit more now: If $0 < R < 1$, then $N^* = K$ is approached without oscillations, since $f'(K) > 0$; if $1 < R < 2$, $N^* = K$ is approached *with* oscillations, since $f'(K) < 0$. ■

Section 5.6 Problems

■ 5.6.1

1. Assume a discrete-time population whose size at generation $t + 1$ is related to the size of the population at generation t by

$$N_{t+1} = (1.03)N_t, \quad t = 0, 1, 2, \ldots$$

(a) If $N_0 = 10$, how large will the population be at generation $t = 5$?

(b) How many generations will it take for the population size to reach double the size at generation 0?

2. Suppose a discrete-time population evolves according to

$$N_{t+1} = (0.9)N_t, \quad t = 0, 1, 2\ldots$$

(a) If $N_0 = 50$, how large will the population be at generation $t = 6$?

(b) After how many generations will the size of the population be one-quarter of its original size?

(c) What will happen to the population in the long run—that is, as $t \to \infty$?

3. Assume the discrete-time population model

$$N_{t+1} = bN_t, \quad t = 0, 1, 2\ldots$$

Assume also that the population increases by 2% each generation.

(a) Determine b.

(b) Find the size of the population at generation 10 when $N_0 = 20$.

(c) After how many generations will the population size have doubled?

4. Assume the discrete-time population model

$$N_{t+1} = bN_t, \quad t = 0, 1, 2, \ldots$$

Assume also that the population decreases by 3% each generation.

(a) Determine b.

(b) Find the size of the population at generation 10 when $N_0 = 50$.

(c) How long will it take until the population is one-half its original size?

5. Assume the discrete-time population model

$$N_{t+1} = bN_t, \quad t = 0, 1, 2, \ldots$$

Assume that the population increases by x% each generation.

(a) Determine b.

(b) After how many generations will the population size have doubled? Compute the doubling time for $x = 0.1, 0.5, 1, 2, 5$, and 10.

6. (a) Find all equilibria of

$$N_{t+1} = 1.3N_t, \quad t = 0, 1, 2, \ldots$$

(b) Use cobwebbing to determine the stability of the equilibria you found in (a).

7. (a) Find all equilibria of

$$N_{t+1} = 0.9N_t, \quad t = 0, 1, 2, \ldots$$

(b) Use cobwebbing to determine the stability of the equilibria you found in (a).

8. (a) Find all equilibria of

$$N_{t+1} = N_t, \quad t = 0, 1, 2, \ldots$$

(b) How will the population size N_t change over time, starting at time 0 with N_0?

■ 5.6.2

9. Use the stability criterion to characterize the stability of the equilibria of

$$x_{t+1} = \frac{2}{3} - \frac{2}{3}x_t^2, \quad t = 0, 1, 2, \ldots$$

10. Use the stability criterion to characterize the stability of the equilibria of

$$x_{t+1} = \frac{3}{5}x_t^2 - \frac{2}{5}, \quad t = 0, 1, 2, \ldots$$

11. Use the stability criterion to characterize the stability of the equilibria of

$$x_{t+1} = \frac{x_t}{0.5 + x_t}, \quad t = 0, 1, 2, \ldots$$

12. Use the stability criterion to characterize the stability of the equilibria of

$$x_{t+1} = \frac{x_t}{0.3 + x_t}, \quad t = 0, 1, 2, \ldots$$

13. **(a)** Use the stability criterion to characterize the stability of the equilibria of

$$x_{t+1} = \frac{5x_t^2}{4 + x_t^2}, \quad t = 0, 1, 2, \ldots$$

(b) Use cobwebbing to decide to which value x_t converges as $t \to \infty$ if **(i)** $x_0 = 0.5$ and **(ii)** $x_0 = 2$.

14. **(a)** Use the stability criterion to characterize the stability of the equilibria of

$$x_{t+1} = \frac{10x_t^2}{9 + x_t^2}, \quad t = 0, 1, 2, \ldots$$

(b) Use cobwebbing to decide to which value x_t converges as $t \to \infty$ if (i) $x_0 = 0.5$ and (ii) $x_0 = 3$.

■ 5.6.3

15. Ricker's curve is given by

$$R(P) = \alpha P e^{-\beta P}$$

for $P \geq 0$, where P denotes the size of the parental stock and $R(P)$ the number of recruits. The parameters α and β are positive constants.

(a) Show that $R(0) = 0$ and $R(P) > 0$ for $P > 0$.

(b) Find

$$\lim_{P \to \infty} R(P)$$

(c) For what size of the parental stock is the number of recruits maximal?

(d) Does $R(P)$ have inflection points? If so, find them.

(e) Sketch the graph of $f(x)$ when $\alpha = 2$ and $\beta = 1/2$.

16. Suppose that the size of a fish population at generation t is given by

$$N_{t+1} = 1.5N_t e^{-0.001N_t}$$

for $t = 0, 1, 2, \ldots$.

(a) Assume that $N_0 = 100$. Find the size of the fish population at generation t for $t = 1, 2, \ldots, 20$.

(b) Assume that $N_0 = 800$. Find the size of the fish population at generation t for $t = 1, 2, \ldots, 20$.

(c) Determine all fixed points. On the basis of your computations in (a) and (b), make a guess as to what will happen to the population in the long run, starting from **(i)** $N_0 = 100$ and **(ii)** $N_0 = 800$.

(d) Use the cobwebbing method to illustrate your answer in (a).

(e) Explain why the dynamical system converges to the nontrivial fixed point.

17. Suppose that the size of a fish population at generation t is given by

$$N_{t+1} = 10N_t e^{-0.01N_t}$$

for $t = 0, 1, 2, \ldots$.

(a) Assume that $N_0 = 100$. Find the size of the fish population at generation t for $t = 1, 2, \ldots, 20$.

(b) Show that if $N_0 = 100 \ln 10$, then $N_t = 100 \ln 10$ for $t = 1, 2, 3, \ldots$; that is, show that $N^* = 100 \ln 10$ is a nontrivial fixed point, or equilibrium. How would you find N^*? Are there any other equilibria?

(c) On the basis of your computations in (a), make a prediction about the long-term behavior of the fish population when $N_0 = 100$. How does your answer compare with that in (b)?

(d) Use the cobwebbing method to illustrate your answer in (c).

In Problems 18–20, consider the following discrete-time dynamical system, which is called the discrete logistic model and which models the size of a population over time:

$$N_{t+1} = N_t \left[1 + R\left(1 - \frac{N_t}{100}\right)\right]$$

for $t = 0, 1, 2, \ldots$.

18. **(a)** Find all equilibria when $R = 0.5$.

(b) Investigate the system when $N_0 = 10$ and describe what you see.

19. **(a)** Find all equilibria when $R = 1.5$.

(b) Investigate the system when $N_0 = 10$ and describe what you see.

20. **(a)** Find all equilibria when $R = 2.5$.

(b) Investigate the system when $N_0 = 10$ and describe what you see.

In Problems 21–22, we investigate the canonical discrete-time logistic growth model

$$x_{t+1} = rx_t(1 - x_t)$$

for $t = 0, 1, 2, \ldots$.

21. Show that for $r > 1$, there are two fixed points. For which values of r is the nonzero fixed point locally stable?

22. Use a calculator or a spreadsheet to simulate the canonical discrete-time logistic growth model with $x_0 = 0.1$ for $t = 0, 1, 2, \ldots, 100$, and describe the behavior when

(a) $r = 3.20$ **(b)** $r = 3.52$ **(c)** $r = 3.80$
(d) $r = 3.83$ **(e)** $r = 3.828$

In Problems 23–25, we consider density-dependent population growth models of the form

$$N_{t+1} = R(N_t)N_t$$

The function $R(N)$ describes the per capita growth. Various forms have been considered. For each function $R(N)$, find all nontrivial fixed points N^* (i.e., $N^* > 0$) and determine the stability as a function of the parameter values. We assume that the function parameters are $r > 0$, $K > 0$, and $\gamma > 1$.

23. $R(N) = rN^{1-\gamma}$ **24.** $R(N) = \dfrac{r}{1 + N/K}$

25. $R(N) = e^{r(1-N/K)}$

compute x_1, we find that

$$x_1 = (-0.7) - \frac{(-0.7)^3 - (-0.7)}{4(-0.7)^2 - 2} = 8.225$$

Successive values are collected in the following list:

$$x_2 = 6.184 \qquad x_7 = 1.613$$
$$x_3 = 4.659 \qquad x_8 = 1.306$$
$$x_4 = 3.521 \qquad x_9 = 1.115$$
$$x_5 = 2.678 \qquad x_{10} = 1.024$$
$$x_6 = 2.059 \qquad x_{11} = 1.001$$

We conclude that the method converges to the root $r = 1$. The situation is illustrated in Figure 5.78. ■

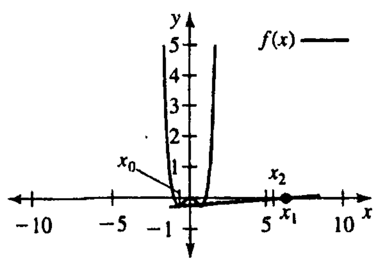

Figure 5.78 The graph of $f(x)$ in Example 5 together with the first two approximations.

Section 5.7 Problems

1. Use the Newton–Raphson method to find a numerical approximation to the solution of

$$x^2 - 7 = 0$$

that is correct to six decimal places.

2. Use the Newton–Raphson method to find a numerical approximation to the solution of

$$e^{-x} = x$$

that is correct to six decimal places.

3. Use the Newton–Raphson method to find a numerical approximation to the solution of

$$x^2 + \ln x = 0$$

that is correct to six decimal places.

4. The equation

$$x^2 - 5 = 0$$

has two solutions. Use the Newton–Raphson method to approximate the two solutions.

5. Use the Newton–Raphson method to solve the equation

$$\sin x = \frac{1}{2}x$$

in the interval $(0, \pi)$.

6. Let

$$f(x) = \begin{cases} \sqrt{x-1} & \text{for } x \geq 1 \\ -\sqrt{1-x} & \text{for } x \leq 1 \end{cases}$$

(a) Show that if you use the Newton–Raphson method to solve $f(x) = 0$, then the following statement holds: If $x_0 = 1 + h$, then $x_1 = 1 - h$, and if $x_0 = 1 - h$, then $x_1 = 1 + h$.

(b) Does the Newton–Raphson method converge? Use a graph to explain what happens.

7. In Example 4, we discussed the case of finding the root of $x^{1/3} = 0$.

(a) Given x_0, find a formula for $|x_n|$.

(b) Find

$$\lim_{n\to\infty} |x_n|$$

(c) Graph $f(x) = x^{1/3}$ and illustrate what happens when you apply the Newton–Raphson method.

8. In Example 5, we considered the equation

$$x^4 - x^2 = 0$$

(a) What happens if you choose

$$x_0 = -\frac{1}{2}\sqrt{2}$$

in the Newton–Raphson method? Give a graphical illustration.

(b) Repeat the procedure in (a) for $x_0 = -0.71$, and compare your result with the result we obtained in Example 5 when $x_0 = -0.70$. Give a graphical illustration and explain it in words. What happens when $x_0 = -0.6$? (This is an example in which small changes in the initial value can drastically change the outcome.)

9. Use the Newton–Raphson method to find a numerical approximation to the solution of

$$x^2 - 16 = 0$$

when your initial guess is (a) $x_0 = 3$ and (b) $x_0 = 4$.

10. Suppose that you wish to use the Newton–Raphson method to solve

$$f(x) = 0$$

numerically. It just so happens that your initial guess x_0 satisfies $f(x_0) = 0$. What happens to subsequent iterations? Give a graphical illustration of your results. [Assume that $f'(x_0) \neq 0$.]

Section 5.8 Problems

In Problems 1–40, find the general antiderivative of the given function.

1. $f(x) = 4x^2 - x$
2. $f(x) = 2 - 5x^2$
3. $f(x) = x^2 + 3x - 4$
4. $f(x) = 3x^2 - x^4$
5. $f(x) = x^4 - 3x^2 + 1$
6. $f(x) = 2x^3 + x^2 - 5x$
7. $f(x) = 4x^3 - 2x + 3$
8. $f(x) = x - 2x^2 - 3x^3 - 4x^4$
9. $f(x) = 1 + \frac{1}{x} + \frac{1}{x^2}$
10. $f(x) = x^2 - \frac{2}{x^2} + \frac{3}{x^3}$
11. $f(x) = 1 - \frac{1}{x^2}$
12. $f(x) = x^3 - \frac{1}{x^3}$
13. $f(x) = \frac{1}{1+x}$
14. $f(x) = \frac{x}{1+x}$
15. $f(x) = 5x^4 + \frac{5}{x^4}$
16. $f(x) = x^7 + \frac{1}{x^7}$
17. $f(x) = \frac{1}{1+2x}$
18. $f(x) = \frac{1}{1+3x}$
19. $f(x) = e^{-3x}$
20. $f(x) = e^{x/2} + e^{-x/2}$
21. $f(x) = 2e^{2x}$
22. $f(x) = -3e^{-4x}$
23. $f(x) = \frac{1}{e^{2x}}$
24. $f(x) = \frac{3}{e^{-x}}$
25. $f(x) = \sin(2x)$
26. $f(x) = \cos(3x)$
27. $f(x) = \sin\left(\frac{x}{3}\right) + \cos\left(\frac{x}{3}\right)$
28. $f(x) = \cos\left(\frac{x}{5}\right) - \sin\left(\frac{x}{5}\right)$
29. $f(x) = 2\sin\left(\frac{\pi}{2}x\right) - 3\cos\left(\frac{\pi}{2}x\right)$
30. $f(x) = -3\sin\left(\frac{\pi}{3}x\right) + 4\cos\left(-\frac{\pi}{4}x\right)$
31. $f(x) = \sec^2(2x)$
32. $f(x) = \sec^2(-4x)$
33. $f(x) = \sec^2\left(\frac{x}{3}\right)$
34. $f(x) = \sec^2\left(-\frac{x}{4}\right)$
35. $f(x) = \frac{\sec x + \cos x}{\cos x}$
36. $f(x) = \sin^2 x + \cos^2 x$
37. $f(x) = x^{-7} + 3x^5 + \sin(2x)$
38. $f(x) = 2e^{-3x} + \sec^2\left(-\frac{x}{2}\right)$
39. $f(x) = \sec^2(3x-1) + \frac{x^2-3}{x}$
40. $f(x) = 5e^{3x} - \sec^2(x-3)$

In Problems 41–46, assume that a is a positive constant. Find the general antiderivative of the given function.

41. $f(x) = \frac{e^{(a+1)x}}{a}$
42. $f(x) = \sin^2(a^2x + 1)$
43. $f(x) = \frac{1}{ax+3}$
44. $f(x) = \frac{a}{a+x}$
45. $f(x) = x^{a+2} - a^{x+2}$
46. $f(x) = \frac{e^{-ax} + e^{ax}}{2a}$

In Problems 47–58, find the general solution of the differential equation.

47. $\frac{dy}{dx} = \frac{2}{x} - x,\ x > 0$
48. $\frac{dy}{dx} = \frac{2}{x^3} - x^3,\ x > 0$
49. $\frac{dy}{dx} = x(1+x),\ x > 0$
50. $\frac{dy}{dx} = e^{-4x},\ x > 0$
51. $\frac{dy}{dt} = t(1-t),\ t \ge 0$
52. $\frac{dy}{dt} = t^2(1-t^2),\ t \ge 0$
53. $\frac{dy}{dt} = e^{-t/2},\ t \ge 0$
54. $\frac{dy}{dt} = 1 - e^{-3t},\ t \ge 0$
55. $\frac{dy}{ds} = \sin(\pi s),\ 0 \le s \le 1$
56. $\frac{dy}{ds} = \cos(2\pi s),\ 0 \le s \le 1$
57. $\frac{dy}{dx} = \sec^2\left(\frac{x}{2}\right),\ -1 < x < 1$
58. $\frac{dy}{dx} = 1 + \sec^2\left(\frac{x}{4}\right),\ -1 < x < 1$

In Problems 59–72, solve the initial-value problem.

59. $\frac{dy}{dx} = 3x^2$, for $x \ge 0$ with $y = 1$ when $x = 0$
60. $\frac{dy}{dx} = \frac{x^2}{3}$, for $x \ge 0$ with $y = 2$ when $x = 0$
61. $\frac{dy}{dx} = 2\sqrt{x}$, for $x \ge 0$ with $y = 2$ when $x = 1$
62. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, for $x \ge 1$ with $y = 3$ when $x = 4$
63. $\frac{dN}{dt} = \frac{1}{t}$, for $t \ge 1$ with $N(1) = 10$
64. $\frac{dN}{dt} = \frac{t}{t+2}$, for $t \ge 0$ with $N(0) = 2$
65. $\frac{dW}{dt} = e^t$, for $t \ge 0$ with $W(0) = 1$
66. $\frac{dW}{dt} = e^{-3t}$, for $t \ge 0$ with $W(0) = 2$
67. $\frac{dW}{dt} = e^{-3t}$, for $t \ge 0$ with $W(0) = 2/3$
68. $\frac{dW}{dt} = e^{-5t}$, for $t \ge 0$ with $W(0) = 1$
69. $\frac{dT}{dt} = \sin(\pi t)$, for $t \ge 0$ with $T(0) = 3$
70. $\frac{dT}{dt} = \cos(\pi t)$, for $t \ge 0$ with $T(0) = 3$
71. $\frac{dy}{dx} = \frac{e^{-x} + e^x}{2}$, for $x \ge 0$ with $y = 0$ when $x = 0$
72. $\frac{dN}{dt} = t^{-1/3}$, for $t > 0$ with $N(0) = 60$
73. Suppose that the length of a certain organism at age x is given by $L(x)$, which satisfies the differential equation

$$\frac{dL}{dx} = e^{-0.1x}, \quad x \ge 0$$

Find $L(x)$ if the limiting length L_∞ is given by

$$L_\infty = \lim_{x\to\infty} L(x) = 25$$

How big is the organism at age $x = 0$?

74. Fish are indeterminate growers; that is, their length $L(x)$ increases with age x throughout their lifetime. If we plot the growth rate dL/dx versus age x on semilog paper, a straight line with negative slope results. Set up a differential equation that relates growth rate and age. Solve this equation under the assumption that $L(0) = 5$, $L(1) = 10$, and

$$\lim_{x\to\infty} L(x) = 20$$

Graph the solution $L(x)$ as a function of x.

75. An object is dropped from a height of 100 ft. Its acceleration is 32 ft/s^2. When will the object hit the ground, and what will its speed be at impact?

76. Suppose that the growth rate of a population at time t undergoes seasonal fluctuations according to

$$\frac{dN}{dt} = 3\sin(2\pi t)$$

where t is measured in years and $N(t)$ denotes the size of the population at time t. If $N(0) = 10$ (measured in thousands), find an expression for $N(t)$. How are the seasonal fluctuations in the growth rate reflected in the population size?

77. Suppose that the amount of water contained in a plant at time t is denoted by $V(t)$. Due to evaporation, $V(t)$ changes over time. Suppose that the change in volume at time t, measured over a 24-hour period, is proportional to $t(24 - t)$, measured in grams per hour. To offset the water loss, you water the plant at a constant rate of 4 grams of water per hour.

(a) Explain why

$$\frac{dV}{dt} = -at(24 - t) + 4$$

$0 \le t \le 24$, for some positive constant a, describes this situation.

(b) Determine the constant a for which the net water loss over a 24-hour period is equal to 0.

Chapter 5 Key Terms

Discuss the following definitions and concepts:

1. Global or absolute extrema
2. Local or relative extrema: local minimum and local maximum
3. The extreme-value theorem
4. Fermat's theorem
5. Mean-value theorem
6. Rolle's theorem
7. Increasing and decreasing function
8. Monotonicity and the first derivative
9. Concavity: concave up and concave down
10. Concavity and the second derivative
11. Diminishing return
12. Candidates for local extrema
13. Monotonicity and local extrema
14. The second-derivative test for local extrema
15. Inflection points
16. Inflection points and the second derivative
17. Asymptotes: horizontal, vertical, and oblique
18. Using calculus to graph functions
19. L'Hospital's rule
20. Dynamical systems: cobwebbing
21. Stability of equilibria
22. Newton–Raphson method for finding roots
23. Antiderivative

Chapter 5 Review Problems

1. Suppose that

$$f(x) = xe^{-x}, \quad x \ge 0$$

(a) Show that $f(0) = 0$, $f(x) > 0$ for $x > 0$, and

$$\lim_{x\to\infty} f(x) = 0$$

(b) Find local and absolute extrema.

(c) Find inflection points.

(d) Use the foregoing information to graph $f(x)$.

2. Suppose that

$$f(x) = x\ln x, \quad x > 0$$

(a) Define $f(x)$ at $x = 0$ so that $f(x)$ is continuous for all $x \ge 0$.

(b) Find extrema and inflection points.

(c) Graph $f(x)$.

3. In Review Problem 17 of Chapter 2 we introduced the hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad x \in \mathbf{R}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad x \in \mathbf{R}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad x \in \mathbf{R}$$

(a) Show that $f(x) = \tanh x$, $x \in \mathbf{R}$, is a strictly increasing function on $\mathbf{R}$. Evaluate

$$\lim_{x\to-\infty} \tanh x$$

and

$$\lim_{x\to\infty} \tanh x$$

(b) Use your results in (a) to explain why $f(x) = \tanh x$, $x \in \mathbf{R}$, is invertible, and show that its inverse function $f^{-1}(x) = \tanh^{-1} x$ is given by

$$f^{-1}(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$$

What is the domain of $f^{-1}(x)$?

(c) Show that

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{1-x^2}$$

(d) Use your result in (c) and the facts that

$$\tanh x = \frac{\sinh x}{\cosh x}$$

and

$$\cosh^2 x - \sinh^2 x = 1$$

to show that

$$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x}$$

4. Let

$$f(x) = \frac{x}{1+e^{-x}}, \quad x \in \mathbf{R}$$

(a) Show that $y = 0$ is a horizontal asymptote as $x \to -\infty$.

(b) Show that $y = x$ is an oblique asymptote as $x \to +\infty$.

(c) Show that

$$f'(x) = \frac{1+e^{-x}(1+x)}{(1+e^{-x})^2}$$

(d) Use your result in (c) to show that $f(x)$ has exactly one local extremum at $x = c$, where c satisfies the equation

$$1 + c + e^c = 0$$

[*Hint*: Use your result in (c) to show that $f'(x) = 0$ if and only if $1 + e^{-x}(1+x) = 0$. Let $g(x) = 1 + e^{-x}(1+x)$. Show that $g(x)$ is strictly increasing for $x < 0$, that $g(0) > 0$, and $g(-2) < 0$. This implies that $g(x) = 0$ has exactly one solution on $(-2, 0)$. Since $g(-2) < 0$ and $g(x)$ is strictly increasing for $x < 0$, there are no solutions of $g(x) = 0$ for $x < -2$. Furthermore, $g(x) > 0$ for $x > 0$; hence, there are no solutions of $g(x) = 0$ for $x > 0$.]

(e) The equation $1 + c + e^c = 0$ can be solved for c only numerically. With the help of a calculator, find a numerical approximation to c. [*Hint*: From (d), you know that $c \in (-2, 0)$.]

(f) Show that $f(x) < 0$ for $x < 0$. [This implies that, for $x < 0$, the graph of $f(x)$ is below the horizontal asymptote $y = 0$.]

(g) Show that $x - f(x) > 0$ for $x > 0$. [This implies that, for $x > 0$, the graph of $f(x)$ is below the oblique asymptote $y = x$.]

(h) Use your results in (a)–(g) and the fact that $f(0) = 0$ and $f'(0) = \frac{1}{2}$ to sketch the graph of $f(x)$.

5. Recruitment Model **Ricker's curve** describes the relationship between the size of the parental stock of some fish and the number of recruits. If we denote the size of the parental stock by P and the number of recruits by R, then Ricker's curve is given by

$$R(P) = \alpha P e^{-\beta P} \quad \text{for } P \geq 0$$

where α and β are positive constants. [Note that $R(0) = 0$; that is, without parents there are no offspring. Furthermore, $R(P) > 0$ when $P > 0$.]

We are interested in the size P of the parental stock that maximizes the number $R(P)$ of recruits. Since $R(P)$ is differentiable, we can use its first derivative to solve this problem.

(a) Use the product rule to show that, for $P > 0$,

$$R'(P) = \alpha e^{-\beta P}(1 - \beta P)$$
$$R''(P) = -\alpha\beta e^{-\beta P}(2 - \beta P)$$

(b) Show that $R'(P) = 0$ if $P = 1/\beta$ and that $R''(1/\beta) < 0$. This shows that $R(P)$ has a *local* maximum at $P = \frac{1}{\beta}$. Show that $R(1/\beta) = \frac{\alpha}{\beta}e^{-1} > 0$.

(c) To find the global maximum, you need to check $R(0)$ and $\lim_{P\to\infty} R(P)$. Show that

$$R(0) = 0 \quad \text{and} \quad \lim_{P\to\infty} R(P) = 0$$

and that this implies that there is a global maximum at $P = 1/\beta$.

(d) Show that $R(P)$ has an inflection point at $P = 2/\beta$.

(e) Sketch the graph of $R(P)$ for $\alpha = 2$ and $\beta = 1$.

6. Gompertz Growth Model The *Gompertz growth curve* is sometimes used to study the growth of populations. Its properties are quite similar to the properties of the logistic growth curve. The Gompertz growth curve is given by

$$N(t) = K \exp[-ae^{-bt}]$$

for $t \geq 0$, where K and b are positive constants.

(a) Show that $N(0) = Ke^{-a}$ and, hence,

$$a = \ln \frac{K}{N_0}$$

if $N_0 = N(0)$.

(b) Show that $y = K$ is a horizontal asymptote and that $N(t) < K$ if $N_0 < K$, $N(t) = K$ if $N_0 = K$, and $N(t) > K$ if $N_0 > K$.

(c) Show that

$$\frac{dN}{dt} = bN(\ln K - \ln N)$$

and

$$\frac{d^2N}{dt^2} = b\frac{dN}{dt}[\ln K - \ln N - 1]$$

(d) Use your results in (b) and (c) to show that $N(t)$ is strictly increasing if $N_0 < K$ and strictly decreasing if $N_0 > K$.

(e) When does $N(t)$, $t \geq 0$, have an inflection point? Discuss its concavity.

(f) Graph $N(t)$ when $K = 100$ and $b = 1$ if **(i)** $N_0 = 20$, **(ii)** $N_0 = 70$, and **(iii)** $N_0 = 150$, and compare your graphs with your answers in (b)–(e).

7. Monod Growth Model The Monod growth curve is given by

$$f(x) = \frac{cx}{k+x}$$

for $x \geq 0$, where c and k are positive constants. The equation can be used to describe the specific growth rate of a species as a function of a resource level x.

(a) Show that $y = c$ is a horizontal asymptote for $x \to \infty$. The constant c is called the *saturation value*.

(b) Show that $f(x)$, $x \geq 0$, is strictly increasing and concave down. Explain why this implies that the saturation value is equal to the maximal specific growth rate.

(c) Show that if $x = k$, then $f(x)$ is equal to half the saturation value. (For this reason, the constant k is called the *half-saturation constant*.)

(d) Sketch a graph of $f(x)$ for $k = 2$ and $c = 5$, clearly marking the saturation value and the half-saturation constant. Compare this graph with one where $k = 3$ and $c = 5$.

(e) Without graphing the three curves, explain how you can use the saturation value and the half-saturation constant to decide quickly that

$$\frac{10x}{3+x} > \frac{10x}{5+x} > \frac{8x}{5+x}$$

for $x \geq 0$.

8. Logistic Growth The logistic growth curve is given by

$$N(t) = \frac{K}{1 + (\frac{K}{N_0} - 1)e^{-rt}}$$

for $t \geq 0$, where K, N_0, and r are positive constants and $N(t)$ denotes the population size at time t.

(a) Show that $N(0) = N_0$ and that $y = K$ is a horizontal asymptote as $t \to \infty$.

(b) Show that $N(t) < K$ if $N_0 < K$, $N(t) = K$ if $N_0 = K$, and $N(t) > K$ if $N_0 > K$.

(c) Show that

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

and

$$\frac{d^2N}{dt^2} = r\frac{dN}{dt}\left(1 - \frac{2N}{K}\right)$$

(d) Use your results in (b) and (c) to show that $N(t)$ is strictly increasing if $N_0 < K$ and strictly decreasing if $N_0 > K$.

(e) Show that if $N_0 < K/2$, then $N(t)$, $t \geq 0$, has exactly one inflection point $(t^*, N(t^*))$, with $t^* > 0$ and

$$N(t^*) = \frac{K}{2},$$

(i.e., half the carrying capacity). What happens if $K/2 < N_0 < K$? What if $N_0 > K$? Where is the function $N(t)$, $t \geq 0$, concave up, and where is it concave down?

(f) Sketch the graphs of $N(t)$ for $t \geq 0$ when
(i) $K = 100, N_0 = 10, r = 1$
(ii) $K = 100, N_0 = 70, r = 1$
(iii) $K = 100, N_0 = 150, r = 1$
Sketch the respective horizontal asymptotes. Mark the inflection point clearly if it exists.

9. Genetics A population is said to be in Hardy–Weinberg equilibrium, with respect to a single gene with two alleles A and a, if the three genotypes AA, Aa, and aa have respective frequencies $p_{AA} = \theta^2$, $p_{Aa} = 2\theta(1-\theta)$, and $p_{aa} = (1-\theta)^2$ for some $\theta \in [0, 1]$. Suppose that we take a random sample of size n from a population. We can show that the probability of observing n_1 individuals of type AA, n_2 individuals of type Aa, and n_3 individuals of type aa is given by

$$\frac{n!}{n_1!\,n_2!\,n_3!} p_{AA}^{n_1} p_{Aa}^{n_2} p_{aa}^{n_3}$$

where $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ (read "n factorial"). Here, $n_1 + n_2 + n_3 = n$. This probability depends on θ. There is a method, called the *maximum likelihood method*, that can be used to estimate θ. The principle is simple: We find the value of θ that maximizes the probability of the observed data. Since the coefficient

$$\frac{n!}{n_1!\,n_2!\,n_3!}$$

does not depend on θ, we need only maximize

$$L(\theta) = p_{AA}^{n_1} p_{Aa}^{n_2} p_{aa}^{n_3}$$

(a) Suppose $n_1 = 8$, $n_2 = 6$, and $n_3 = 3$. Compute $L(\theta)$.

(b) Show that if $L(\theta)$ is maximal for $\theta = \hat{\theta}$ (read "theta hat"), then $\ln L(\theta)$ is also maximal for $\theta = \hat{\theta}$.

(c) Use your result in (b) to find the value $\hat{\theta}$ that maximizes $L(\theta)$ for the data given in (a). The number $\hat{\theta}$ is the maximum likelihood estimate.

10. Cell Volume Suppose the volume of a cell is increasing at a constant rate of 10^{-12} cm^3/s.

(a) If $V(t)$ denotes the cell volume at time t, set up an initial-value problem that describes this situation if the initial volume is 10^{-10} cm^3.

(b) Solve the initial-value problem given in (a), and determine the volume of the cell after 10 seconds.

11. Drug Concentration Suppose the concentration $c(t)$ of a drug in the bloodstream at time t satisfies

$$\frac{dc}{dt} = -0.1e^{-0.3t}$$

for $t \geq 0$.

(a) Solve the differential equation under the assumption that there will eventually be no trace of the drug in the blood.

(b) How long does it take until the concentration reaches half its initial value?

12. Resource-Limited Growth Sterner (1997) investigated the effect of food quality on zooplankton dynamics. In his model, zooplankton may be limited by either carbon (C) or phosphorus (P). He argued that when food quantity is low, demand for carbon increases relative to demand for phosphorus in order to to satisfy basic metabolic requirements and that there should be a curve separating C- and P-limited growth when food quantity C_F (measured in amount of carbon per liter) is graphed as a function of the C:P ratio of the food, $f = C_F:P_F$. He derived the following equation for the curve separating the two regions:

$$C_F = \frac{m}{a_C g - \frac{C_Z a_P g}{P_Z f}}$$

Here, m denotes the respiration rate, g the ingestion rate, and a_C (a_P) the assimilation rate of carbon (phosphorus). C_Z and P_Z are, respectively, the carbon and the phosphorus content of the zooplankton.

(a) Show that the graph of $y = C_F(f)$ approaches the horizontal line $y = \frac{m}{a_C g}$ as $f \to \infty$.

(b) The graph of $C_F(f)$ has a vertical asymptote. Let $f = C_F:P_F$ (the C:P ratio of the food). Show that the vertical asymptote is at

$$\frac{C_F}{P_F} = \frac{C_Z}{P_Z}\frac{a_P}{a_C}$$

(c) Sketch a graph of $C_F(f)$ as a function of f.

(d) The graph of $C_F(f)$ separates C-limited (below the curve) from P-limited (above the curve) growth. Explain why this graph indicates that when food quantity is low, the demand for carbon relative to phosphorus increases.

13. Velocity and Distance Neglecting air resistance, the height (in meters) of an object thrown vertically from the ground with initial velocity v_0 is given by

$$h(t) = v_0 t - \frac{1}{2}gt^2$$

where $g = 9.81\text{m/s}^2$ is the earth's gravitational constant and t is the time (in seconds) elapsed since the object was released

(a) Find the time at which the object reaches its maximum height.

(b) Find the maximum height.

(c) Find the velocity of the object at the time it reaches its maximum height.

(d) At what time $t > 0$ will the object reach the initial height again?