Math 17A
Kouba
Recursions, Sequences, Fixed Points, and Limits

EXAMPLE: The following recursions and initial values determine a sequence. Find \( a_n \) for \( n = 1, 2, 3, 4, 5 \).

1.) \( a_{n+1} = 2a_n + 3, \ a_0 = -1 \)
   \( a_1 = 2a_0 + 3 = 2(-1) + 3 = 1, \)
   \( a_2 = 2a_1 + 3 = 2(1) + 3 = 5, \)
   \( a_3 = 2a_2 + 3 = 2(5) + 3 = 13, \)
   \( a_4 = 2a_3 + 3 = 2(13) + 3 = 29, \)
   \( a_5 = 2a_4 + 3 = 2(29) + 3 = 61. \) Hence \( \lim_{n \to \infty} a_n = \infty \) (DNE).

2.) \( a_{n+1} = 2a_n + 3, \ a_0 = -3 \)
   \( a_1 = 2a_0 + 3 = 2(-3) + 3 = -3, \)
   \( a_2 = 2a_1 + 3 = 2(-3) + 3 = -3, \)
   \( a_3 = 2a_2 + 3 = 2(-3) + 3 = -3, \)
   \( a_4 = 2a_3 + 3 = 2(-3) + 3 = -3, \)
   \( a_5 = 2a_4 + 3 = 2(-3) + 3 = -3. \) Hence \( \lim_{n \to \infty} a_n = -3. \)

DEFINITION: Let \( a_{n+1} = f(a_n), \ a_0 = L \), for \( n = 1, 2, 3, 4, \ldots \) be a recursion and initial value which determines a sequence. The initial value \( L \) is called a fixed point for the recursion if all successive values of \( a_n \) are equal to \( L \), i.e., if \( L = f(L) \).

NOTE:
I.) The number \(-3\) is a fixed point for the previous example.
II.) The initial value is sometimes critical in determining if the sequence converges or diverges.
III.) A fixed point represents a potential limit for the sequence generated by the recursion and its initial value.
IV.) Every limit of an associated sequence is a fixed point for the recursion.

EXAMPLE: Find all fixed points for each recursion.

1.) \( a_{n+1} = (1/2)a_n - (3/4) \)

2.) \( a_{n+1} = \frac{2}{a_n - 1} \)

3.) \( a_{n+1} = \frac{a_n^2}{a_n^2 - 12} \)