Section 4.4

2.) \( f(x) = (4x+5)^3 \) \( \Rightarrow \) \( f'(x) = 3(4x+5)^2 \cdot 4 \)

5.) \( f(x) = (x^2+3)^{\frac{1}{2}} \) \( \Rightarrow \) \( f'(x) = \frac{1}{2} (x^2+3)^{-\frac{1}{2}} \cdot 2x \)

9.) \( f(x) = \frac{1}{(x^3-2)^4} \) \( \Rightarrow \) \( f'(x) = -4 (x^3-2)^{-5} \cdot 3x^2 \)

12.) \( f(x) = \frac{(1-2x^2)^3}{(3-x^2)^2} \) \( \Rightarrow \)

\[
f'(x) = \frac{(3-x^2)^2 \cdot 3(1-2x^2)^2 \cdot (-4x) - (1-2x^2)^3 \cdot 2(3-x^2) \cdot (-2x)}{(3-x^2)^4}
\]

16.) \( g(t) = (t^2 + (t+1)^{\frac{1}{2}})^{\frac{1}{2}} \) \( \Rightarrow \)

\[
g'(t) = \frac{1}{2} (t^2 + (t+1)^{\frac{1}{2}})^{-\frac{1}{2}} \cdot \{2t + \frac{1}{2}(t+1)^{-\frac{1}{2}} \cdot (1)^{\frac{3}{2}}\}
\]

17.) \( g(t) = \left(\frac{t}{t-3}\right)^3 \) \( \Rightarrow \)

\[
g'(t) = 3 \left(\frac{t}{t-3}\right)^2 \cdot \frac{(t-3)(1-t)(1)}{(t-3)^2}
\]

19.) \( f(r) = (r^2 - r)^3 \cdot (r+3r^3)^{-4} \) \( \Rightarrow \)

\[
f'(r) = (r^2 - r)^3 \cdot -4(r+3r^3)^{-5} \cdot (1+9r^2)
\]

\[
+ 3(r^2 - r)^2 \cdot (2r-1) \cdot (r+3r^3)^{-4}
\]

24.) \( f(x) = (2-4x^2)^{\frac{1}{4}} \) \( \Rightarrow \)

\[
f'(x) = \frac{1}{4} (2-4x^2)^{-\frac{3}{4}} \cdot (-8x)
\]
27. \( h(t) = \left(3t + \frac{3}{t}\right)^{3/5} \) \\
\[ h'(t) = \frac{\frac{t}{3} \left(3t + \frac{3}{t}\right)^{3/5}}{t^2} \left\{ 3 + \frac{t(0) - 3(1)}{t^2} \right\} \]

34. a.) \( f'(x) = 2x + 1 \) \\
\[ D f(x^2) = f'(x^2) \cdot D(x^2) \]
\[ = (2(x^2) + 1) \cdot 2x \]
\[ = 2x^2 + 2x \]
\[ \text{Let } x = -1 \rightarrow \]
\[ D f(x^2) = (2(-1)^2 + 1)(-2) = -6 \]

35. b.) \( f'(x) = \frac{1}{x} \) \\
\[ D f(\sqrt{x-1}) = f'(\sqrt{x-1}) \cdot D(\sqrt{x-1}) \]
\[ = \frac{1}{\sqrt{x-1}} \cdot \frac{1}{2} (x-1)^{-1/2} \cdot (1) = \frac{1}{2 \sqrt{x-1}} \cdot \frac{1}{x-1} = \frac{1}{2(x-1)} \]

37. \[ D \left( \frac{f(x)}{g(x)} + 1 \right)^2 = 2 \left( \frac{f(x)}{g(x)} + 1 \right) \cdot \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2} \]

38. \[ D f\left( \frac{1}{g(x)} \right) = f'\left( \frac{1}{g(x)} \right) \cdot \frac{g(x) \cdot (0) - (1) \cdot g'(x)}{(g(x))^2} \]
\[ = f'\left( \frac{1}{g(x)} \right) \cdot \frac{-g'(x)}{(g(x))^2} \]

43. \( y = (1 + (3x^2 - 1)^3)^2 \) \\
\[ y' = 2(1 + (3x^2 - 1)^3) \cdot \left\{ 0 + 3(3x^2 - 1)^2 \cdot (6x) \right\} \]

47. \( x^2 + y^2 = 4 \) \\
\[ D \rightarrow 2x + 2y \cdot y' = 0 \]
\[ 2y' = -2x \Rightarrow y' = \frac{-2x}{2y} \Rightarrow y' = \frac{-x}{y} \]

48.) \[ y = x^2 + 3x \Rightarrow \]
\[ y' = 2x + 3x + (3) \Rightarrow y' - 3xy' = 2x + 3y \Rightarrow \]
\[ (1 - 3x) y' = 2x + 3y \Rightarrow y' = \frac{2x + 3y}{1 - 3x} \]

50.) \[ xy - y^3 = 1 \Rightarrow \]
\[ xy' + (y - 3y^2) y' = 0 \Rightarrow \]
\[ xy' - 3y^2 y' = -y \Rightarrow (x - 3y^2) y' = -y \Rightarrow \]
\[ y' = \frac{-y}{x - 3y^2} \]

51.) \[ x^{\frac{1}{2}} y^{\frac{1}{2}} = x^2 + 1 \Rightarrow \]
\[ x^{\frac{1}{2}} \cdot \frac{1}{2} y^{-\frac{1}{2}} \cdot y' + \frac{1}{2} x^{\frac{1}{2}} y' = 2x \Rightarrow \]
\[ \frac{1}{2} \sqrt{x} \cdot \frac{1}{\sqrt{y}} y' = 2x - \frac{1}{2} \frac{1}{\sqrt{x}} \sqrt{y} \Rightarrow \]
\[ y' = \frac{2x - \frac{1}{2} \sqrt{\frac{y}{x}}}{\frac{1}{2} \sqrt{\frac{x}{y}}} \]

52.) \[ \frac{1}{2xy} - y^3 = 4 \Rightarrow \text{(mult. by } 2xy) \Rightarrow \]
\[ 1 - 2xy^4 = 8xy \Rightarrow \]
\[ 0 - (2x \cdot 4y^3 y' + 2y^4) = 8x \cdot y' + 8y \Rightarrow \]
\[ -8xy^3 y' - 2y^4 = 8xy' + 8y \Rightarrow \text{(mult. by } \frac{1}{2}) \]
\[ -4xy^3 y' - y' = 4xy' + 4y \Rightarrow \]
\[ -y' = 4xy^3 y' + 4xy' \Rightarrow \]
\[ y' (4xy^3 + 4x) = -y^4 - 4y \Rightarrow \]
\[ y' = \frac{-y^4 - 4y}{4xy^3 + 4x} \]

53.) \[ \frac{x}{y} = \frac{y}{x} \rightarrow x^2 = y^2 \quad \text{D} \quad 2x = 2yy' \rightarrow \]
\[ y' = \frac{x}{y} \]

54.) \[ \frac{x}{xy+1} = 2xy \rightarrow x = 2xy(xy+1) \rightarrow \]
\[ x = 2x^2y^2 + 2xy \quad \text{D} \]
\[ 1 = 2x^2y + 4xy^2 + 2xy'y + 2y \rightarrow \]
\[ 1 - 4xy^2 - 2y = (4x^2y + 2x)y' \rightarrow \]
\[ y' = \frac{1 - 4xy^2 - 2y}{4x^2y + 2x} \]

55.) \[ x^2 + y^2 = 25 \quad \text{D} \quad 2x + 2yy' = 0 \rightarrow \]
\[ 2yy' = -2x \rightarrow y' = -\frac{2x}{2y} \rightarrow y' = -\frac{x}{y} \]
\[ \text{at pt. } (4,-3) \rightarrow y' = -\frac{4}{-3} = \frac{4}{3} \quad \]
\[ m = y' = \frac{4}{3} \quad \text{so tangent line is} \]
\[ y - (-3) = \frac{4}{3}(x - 4) \rightarrow y + 3 = \frac{4}{3}x - \frac{16}{3} \rightarrow \]
\[ y = \frac{4}{3}x - \frac{25}{3} \]
60. a.) \[ y^2 = 10x^4 - x^2 \quad \Rightarrow \quad 2yy' = 40x^3 - 2x \quad \Rightarrow \quad y' = \frac{40x^3 - 2x}{2y} \]
\[
y' = \frac{2}{y} (20x^3 - x) \quad \Rightarrow \quad y' = \frac{20x^3 - x}{y} ;
\]
\[ a \times p \times (1, 3) \Rightarrow y' = \frac{20(1)^3 - 1}{3} = \frac{19}{3} \]

b.) \[ x^2 + y^2 = 1 \quad \Rightarrow \quad \frac{dx}{dt} = 2, \quad x = \frac{1}{2} \quad \Rightarrow \]
\[ \left(\frac{1}{2}\right)^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = \frac{3}{4} \quad \Rightarrow \quad y = \pm \frac{\sqrt{3}}{2} \]
\[ (y > 0) \quad \Rightarrow \quad y = \frac{\sqrt{3}}{2} ; \quad \text{then} \quad \frac{dy}{dt} = \frac{\frac{\sqrt{3}}{2}}{\sqrt{3}} \]
\[ \Rightarrow \quad \frac{\sqrt{3}}{2} \cdot \frac{dy}{dt} = -1 \quad \Rightarrow \quad \frac{dy}{dt} = \frac{-2}{\sqrt{3}} \]

63. \[ x^2y = 1 \quad \Rightarrow \quad \frac{dx}{dt} = 3, \quad x = 2 \quad \Rightarrow \quad 4y = 1 \quad \Rightarrow \quad y = \frac{1}{4} ; \quad \text{then} \quad \frac{dy}{dt} = x^2 \cdot \frac{dy}{dt} + 2x \cdot \frac{dx}{dt} \cdot y = 0 \]
\[ \Rightarrow \quad (2)^2 \cdot \frac{dy}{dt} + 2(2)(3)(\frac{1}{4}) = 0 \quad \Rightarrow \quad \frac{dy}{dt} = \frac{-3}{4} \]

65. \[ v = x^3 \quad \Rightarrow \quad \frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} \]
69.) \( V = 25\pi h \) and \( \frac{dV}{dt} = \frac{250\pi}{\min} \cdot \frac{1\text{ m}^3}{1000\text{ ft}^3} \)

\[ \frac{dV}{dt} = \frac{1}{4} \text{ m}^3 / \min. \Rightarrow \frac{dh}{dt} = 25\pi \cdot \frac{dh}{dt} \]

\[ \frac{1}{4} = 25\pi \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{100\pi} \text{ m/\min}. \]

70.)

By similar \( \Delta \)s

\[ \frac{r}{h} = \frac{3}{6} \Rightarrow \frac{r}{h} = \frac{1}{2} \]

\[ V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left(\frac{1}{2} h\right)^2 h \]

\[ V = \frac{1}{12} \pi h^3 \]

\[ \frac{dV}{dt} = 5 \text{ ft}^3 / \text{min}. \]

\[ V = \frac{1}{12} \pi h^3 \Rightarrow \frac{dV}{dt} = \frac{1}{12} \pi \cdot 3h^2 \cdot \frac{dh}{dt} \]

\[ 5 = \frac{\pi}{4} (2)^2 \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{5}{\pi} \text{ ft/\min}. \]
74. \( f(x) = (2x^2 + 4)^3 \) \( \frac{d}{dx} \) \( f(x) = 3(2x^2 + 4)^2 \cdot 4x \\
= 12x(2x^2 + 4)^2 \) \\
\( f'(x) = 12x \cdot 2(2x^2 + 4) \cdot 4x + (12)(2x^2 + 4)^2 \\
= 12(2x^2 + 4)[8x^2 + (2x^2 + 4)] \\
= 12(2x^2 + 4)[10x^2 + 4] \)
80. \[ f(x) = \frac{2x}{x^2 + 1} \]  \[ \rightarrow \]  \[ f'(x) = \frac{(x^2+1)(2) - 2x(2x)}{(x^2 + 1)^2} \]  \[ = \frac{2 - 2x^2}{(x^2 + 1)^2} \]  \[ \rightarrow \]  \[ f''(x) = \frac{(x^2+1)^2(-4x) - (2-2x^2) \cdot 2(x^2+1) \cdot 2x}{(x^2 + 1)^4} \]  \[ = \frac{-4x(x^2+1)[(x^2+1) + (2-2x^2)]}{(x^2 + 1)^4} \]  \[ = \frac{-4x(3-x^2)}{(x^2 + 1)^3} \]

86. a.) \[ S(t) = t^2 - 3t \]  \[ \rightarrow \]  \[ \frac{d}{dt} \]  \[ \text{vel.} \] \[ S'(t) = 2t - 3 \]  \[ \rightarrow \]  \[ S'(1) = 2 - 3 = -1 \]  \[ \frac{d}{dt} \]  \[ \text{acc.} \] \[ S''(t) = 2 \]  \[ \rightarrow \]  \[ S''(1) = 2 \]

b.) \[ S(t) = \sqrt{t^2 + 1} \]  \[ \rightarrow \]  \[ \frac{d}{dt} \]  \[ \text{vel.} \] \[ S'(t) = \frac{1}{2} (t^2 + 1)^{-\frac{1}{2}} \cdot (2t) = \frac{t}{\sqrt{t^2 + 1}} \]  \[ \rightarrow \]  \[ \frac{d}{dt} \]  \[ \text{acc.} \] \[ S''(t) = -t \cdot \frac{1}{2} (t^2 + 1)^{-\frac{3}{2}} \cdot 2t \]  \[ = \frac{-t^2 + 1}{(t^2 + 1)^{3/2}} \]  \[ \rightarrow \]  \[ S''(t) = \frac{1}{(t^2 + 1)^{3/2}} \]  \[ \rightarrow \]  \[ S''(1) = \frac{1}{2^{3/2}} \]
Gravity Problems

1) \( S = 128 \)

\[ S(t) = -16t^2 + V_0t + S_0 \]

\[ S(t) = -16t^2 + 112t + 128 \]

\[ S'(t) = -32t + 112 \]

\[ S'(t) = 0 \]

a.) highest point: \( S'(t) = 0 \)

\[ -32t + 112 = 0 \]

\[ t = \frac{112}{32} = 3.5 \text{ sec.} \]

\[ S(3.5) = -16(3.5)^2 + 112(3.5) + 128 = 324 \text{ ft.} \]

b.) hit ground: \( S(t) = 0 \)

\[ -16t^2 + 112t + 128 = 0 \]

\[ -16(t^2 - 7t - 8) = 0 \]

\[ -16(t - 8)(t + 1) = 0 \]

\[ t = 8 \text{ sec.} \] (not a valid solution)

C.) i.) \( S'(3) = -32(3) + 112 = 16 \text{ ft./sec} \)

ii.) \( S'(4) = -32(4) + 112 = -16 \text{ ft./sec} \)

iii.) \( S'(8) = -32(8) + 112 = -144 \text{ ft./sec} \)

2) \( S = 320 \)

\[ S(t) = -16t^2 + V_0t + S_0 \]

\[ S(t) = -16t^2 - 16t + 320 \]

\[ S'(t) = -32t - 16 \]

\[ S'(t) = 0 \]

a.) hit ground: \( S(t) = 0 \)

\[ -16t^2 - 16t + 320 = 0 \]

\[ -16(t^2 + t - 20) = 0 \]

\[ -16(t - 4)(t + 5) = 0 \]

\[ t = -5 \text{ (not a valid solution)} \]
b.) i.) \( s'(1) = -32(1) - 16 = -48 \text{ ft./sec.} \)
   ii.) \( s'(2) = -32(2) - 16 = -80 \text{ ft./sec.} \)
   iii.) \( s'(4) = -32(4) - 16 = -144 \text{ ft./sec.} \)

3.)

\[
\begin{align*}
&\text{when } \ t = 2 \\
&\text{when } \ t = 2 \\
&s = 0 \\
&s = 0 \\
&\text{and } \ s'(2) = 0 \rightarrow -32(2) + v_0 = 0 \\
&b.) \ v_0 = 64 \text{ ft./sec.}
\end{align*}
\]

a.) \( s(2) = -16(2)^2 + 64(2) = 64 \text{ ft.} \)

b.)

\[
\begin{align*}
&s = 1600 \\
&s' = 0 \\
&s = 0 \\
&-16t^2 + 1600 = 0 \rightarrow 16t^2 = 1600 \rightarrow t^2 = 100 \rightarrow \\
&t = 10 \text{ sec.}
\end{align*}
\]

b.) i.) \( s'(5) = -32(5) = -160 \text{ ft./sec.} \)
   ii.) \( s'(10) = -32(10) = -320 \text{ ft./sec.} \)

\[
\frac{-320 \text{ ft.}}{\text{sec}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft.}} \cdot \frac{3600 \text{ sec.}}{1 \text{ hr.}} \approx -218.2 \text{ mph}
\]
5. \[ s' = 0 \]
   \[ s = 144 \]

\[ s = 0 \]

Let \( t = T \) be the time required to reach the highest point. Then

\[
\begin{align*}
S'(T) &= -32T + V_0 = 0 \\
S(T) &= -16T^2 + V_0 T = 144
\end{align*}
\]

\( V_0 = \frac{32T}{(s=144)} \)

\( -16T^2 + (32T)T = 144 \rightarrow 16T^2 = 144 \rightarrow T^2 = 9 \rightarrow T = 3 \text{ sec.} \)

a.) \( T = 3 \text{ sec} \)

b.) Hit ground: \( s(t) = 0 \rightarrow -16t^2 + 96t = 0 \rightarrow 16t(6-t) = 0 \rightarrow t = 0 \) and \( t = 6 \text{ sec} \).

c.) \( V_0 = 32T = 32(3) = 96 \text{ ft. sec}^{-1} \)

d.) You know.

6.

\[ s' = 0 \]
\[ s = 8000 \]

\[ s(t) = -16t^2 + V_0 t + S_o \rightarrow s(t) = -16t^2 + 8000 \rightarrow s'(t) = -32t \]

\[ s = 1600 \]

\[ s(t) = -16t^2 + 1600 \rightarrow -16t^2 + 8000 = 1600 \]
\[ 16t^2 = 6400 \rightarrow t^2 = 400 \rightarrow t = 20 \text{ sec.} \]

b. \( S'(20) = -32(20) = -640 \text{ ft./sec.} \)

7.

\[
\begin{align*}
S' &= \ ? \quad (S' = V_0) \\
S &= \ ? \quad (S = S_0)
\end{align*}
\]

\[
S(10) = -16t^2 + V_0t + S_0 \\
S'(t) = -32t + V_0
\]

a. \( S'(10) = -400 \rightarrow -32(10) + V_0 = -400 \rightarrow V_0 = -80 \text{ ft./sec.} \)

\[
\begin{align*}
S &= 4000 \\
T &= T
\end{align*}
\]

b. \( S(T) = 4000 \)

\[
\begin{align*}
S(T + 5) &= 2400 \\
S &= 0
\end{align*}
\]

\[
\begin{align*}
-16T^2 - 80T + S_0 &= 4000 \\
-16(T + 5)^2 - 80(T + 5) + S_0 &= 2400
\end{align*}
\]

\[
S_0 = 4000 + 80T + 16T^2 \quad \text{(SUB)} \rightarrow
\]

\[
-16(T^2 + 10T + 25) - 80T - 400 + (4000 + 80T + 16T^2) = 2400 \rightarrow
\]

\[
-16T^2 - 160T - 400 - 80T - 400 + 4000 + 80T + 16T^2 = 2400 \rightarrow
\]

\[
160T = 800 \rightarrow T = 5 \text{ sec.}
\]

\[
S_0 = 4000 + 80(5) + 16(5)^2 \rightarrow
\]

\[
S_0 = 4800 \text{ ft.}
\]
c.) \( s(t) = -16t^2 - 80t + 4800 \)  
\[ \text{strike ground: } s(t) = 0 \rightarrow \]
\[ -16t^2 - 80t + 4800 = 0 \rightarrow \]
\[ -16(t^2 + 5t - 300) = 0 \rightarrow \]
\[ -16(t - 15)(t + 20) = 0 \rightarrow \]
\[ t = 15 \text{ sec.} \quad t = -20 \]

d.) Snapple Peach Ice Tea

8.)  
\[ \text{(dropped) } s'(t) = 0 \]
\[ s = 0 \]
\[ s'(t) = -32t \]

\[ \text{a.) Given } s(5) = 0 \text{ ft. } \rightarrow -16(5)^2 + H = 0 \]
\[ \rightarrow H = 400 \text{ ft.} \]

\[ \text{b.) } s'(1) = -32(1) = -32 \text{ ft./sec.} \]
\[ s'(3) = -32(3) = -96 \text{ ft./sec.} \]

\[ \text{c.) } s'(5) = -32(5) = -160 \text{ ft./sec.} \]
\[ \frac{-160 \text{ ft.}}{\text{sec.}} \times \frac{1 \text{ mi.}}{5280 \text{ ft.}} \times \frac{3600 \text{ sec.}}{1 \text{ hr.}} \approx 109.1 \text{ mph} \]