

Math 17A

Kouba

A Naturally-Occurring Exponential Function

Musical Pitch

The pitch of a musical note is determined by the frequency of the vibration which causes it. Middle C on the piano, for example, corresponds to a vibration of 263 hertz (cycles per second). A note one octave above middle C vibrates at 526 hertz, and a note two octaves above middle C vibrates at 1052 hertz. (See Table 1.7.)

TABLE 1.7 *Pitch of notes above middle C*

| Number, n , of octaves above middle C | Number of hertz $V = f(n)$ |
|---|-------------------------------|
| 0 | 263 |
| 1 | 526 |
| 2 | 1052 |
| 3 | 2104 |
| 4 | 4208 |

TABLE 1.8 *Pitch of notes below middle C*

| n | $V = 263 \cdot 2^n$ |
|-----|--|
| -3 | $263 \cdot 2^{-3} = 263(1/2^3) = 32.875$ |
| -2 | $263 \cdot 2^{-2} = 263(1/2^2) = 65.75$ |
| -1 | $263 \cdot 2^{-1} = 263(1/2) = 131.5$ |
| 0 | $263 \cdot 2^0 = 263$ |

Notice that

$$\frac{526}{263} = 2 \quad \text{and} \quad \frac{1052}{526} = 2 \quad \text{and} \quad \frac{2104}{1052} = 2$$

and so on. In other words, each value of V is twice the value before, so

$$\begin{aligned} f(1) &= 526 = 263 \cdot 2 = 263 \cdot 2^1 \\ f(2) &= 1052 = 526 \cdot 2 = 263 \cdot 2^2 \\ f(3) &= 2104 = 1052 \cdot 2 = 263 \cdot 2^3. \end{aligned}$$

In general

$$V = f(n) = 263 \cdot 2^n.$$

The base 2 represents the fact that as we go up an octave, the frequency of vibrations doubles. Indeed, our ears hear a note as one octave higher than another precisely because it vibrates twice as fast. For the negative values of n in Table 1.8, this function represents the octaves below middle C. The notes on a piano are represented by values of n between -3 and 4, and the human ear finds values of n between -4 and 7 audible.

Although $V = f(n) = 263 \cdot 2^n$ makes sense in musical terms only for certain values of n , values of the function $f(x) = 263 \cdot 2^x$ can be calculated for all real x , and its graph has the typical exponential shape, as can be seen in Figure 1.19. It is concave up, climbing faster and faster as x increases.

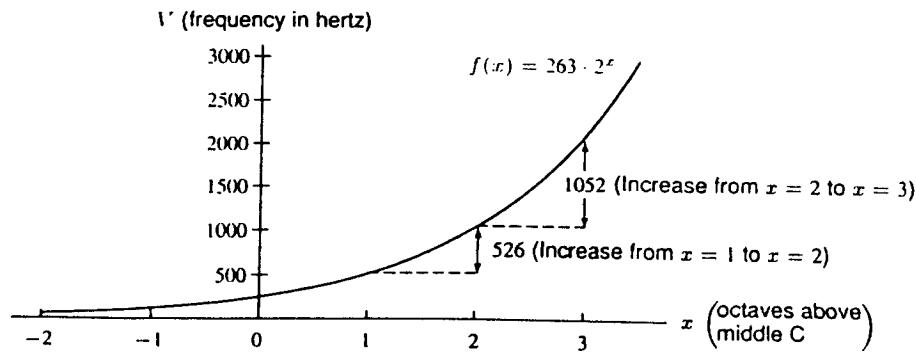


Figure 1.19: Pitch as a function of number of octaves above middle C