Math 17A (Fall 2006)
Kouba
Exam 1

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 6 pages, including the cover page.

6. You may NOT use L'Hopital's Rule on this exam.

7. You may NOT use shortcuts for finding limits to infinity.

8. Using only a calculator to determine limits will receive little or no credit.

9. You will be graded on proper use of limit notation.

10. You have until 8:50 a.m. sharp to finish the exam.

1.) (7 pts. each) Determine the following limits.

a.) $\lim_{x \to 1} \frac{x^2 - 4x + 3}{x^2 - 1}$ (HINT: Factor.)

$$\text{"}\frac{0}{0}\text{"} \quad \lim_{x \to 1} \frac{(\cancel{x-1})(x-3)}{(\cancel{x-1})(x+1)} = \frac{-2}{2} = -1$$

b.) $\lim_{x \to -1} \frac{\sqrt{x+2} - 1}{x+1}$ (HINT: Use a conjugate or factor.)

$$\text{"}\frac{0}{0}\text{"} \quad \lim_{x \to -1} \frac{\sqrt{x+2} - 1}{x+1} \cdot \frac{\sqrt{x+2} + 1}{\sqrt{x+2} + 1} = \lim_{x \to -1} \frac{(x+2) - 1}{(x+1)(\sqrt{x+2} + 1)}$$

$$= \lim_{x \to -1} \frac{\cancel{x+1}}{(\cancel{x+1})(\sqrt{x+2} + 1)} = \frac{1}{1+1} = \frac{1}{2}$$

c.) $\lim_{x \to 2} \frac{\frac{1}{2} - \frac{1}{x}}{2 - x}$ (HINT: Add fractions first.)

$$\text{"}\frac{0}{0}\text{"} \quad \lim_{x \to 2} \frac{x-2}{2x} \cdot \frac{1}{2-x} = \lim_{x \to 2} \frac{-(\cancel{2-x})}{2x \cdot (\cancel{2-x})}$$

$$= -1/4$$

d.) $\lim_{x \to \infty} \frac{4x^3 - 7x}{6x^2 + 5}$ (HINT: Divide by a power of x.)

$$\text{"}\frac{\infty}{\infty}\text{"} \quad \lim_{x \to \infty} \frac{4x^3 - 7x}{6x^2 + 5} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \to \infty} \frac{4x - \frac{7}{x}}{6 + 5/x^2}$$

$$= \frac{\infty - 0}{6 + 0} = \infty$$

e.) $\lim_{x \to -\infty} \frac{2 + e^{-x}}{6 - e^{-x}}$ $\text{"}\frac{\infty}{\infty}\text{"} = \lim_{x \to -\infty} \frac{2 + e^{-x}}{6 - e^{-x}} \cdot \frac{1/e^{-x}}{1/e^{-x}}$

$$= \lim_{x \to -\infty} \frac{\frac{2}{e^{-x}} + 1}{\frac{6}{e^{-x}} - 1} = \frac{\frac{2}{\infty} + 1}{\frac{6}{\infty} - 1} = \frac{0+1}{0-1} = -1$$

2.) Consider the function $f(x) = 7 - \sqrt{6x-2}$.

a.) (3 pts.) Determine the domain of f. Because of $\sqrt{\ }$ we need $6x - 2 \geq 0 \rightarrow 6x \geq 2 \rightarrow x \geq \frac{2}{6} = \frac{1}{3}$

so Domain : $x \geq \frac{1}{3}$

b.) (3 pts.) Determine the range of f. $0 \leq 6x-2 < \infty$ so $0 \leq \sqrt{6x-2} < \infty$ and $-\infty < -\sqrt{6x-2} \leq 0$;

Range : $y \leq 7$

3.) Start with a large piece of paper 1/64 of an inch thick ($n = 0$). Cut the paper in half and place the pieces on top of each other to form a new stack ($n = 1$). Cut the new stack in half and place the pieces on top of each other to form a new stack ($n = 2$). Let a_n be the thickness in inches of the stack after n cuts.

a.) (3 pts.) State the initial value a_0 and give a recursion for a_n for $n = 0, 1, 2, 3, \cdots$.

$a_0 = \frac{1}{64}''$, $a_{n+1} = 2a_n$ for $n = 0, 1, 2, \ldots$

b.) (7 pts.) Determine a_1, a_2, a_3, and a_4 and determine an exponential growth formula for a_n for $n = 0, 1, 2, 3, \cdots$.

$a_1 = \frac{1}{64} \cdot 2$,

$a_2 = \frac{1}{64} \cdot 4 = \frac{1}{64} \cdot 2^2$,

$a_3 = \frac{1}{64} \cdot 8 = \frac{1}{64} \cdot 2^3$,

$a_4 = \frac{1}{64} \cdot 16 = \frac{1}{64} \cdot 2^4$, ...

$\rightarrow a_n = \frac{1}{64} \cdot 2^n$ for $n = 0, 1, 2, \ldots$

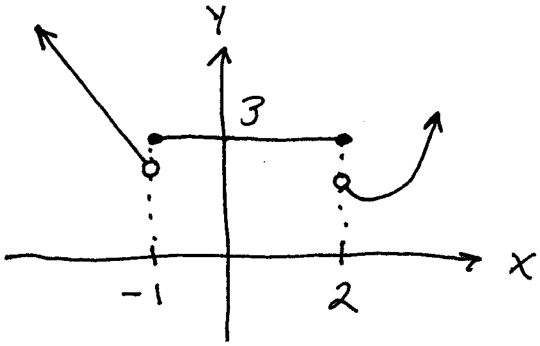
c.) (5 pts. extra credit) How thick is the stack after the 28th cut ? Write your answer in miles (1 mile = 5280 feet) !

$a_{28} = \left(\frac{1}{64} \cdot 2^{28} \text{ in.}\right)\left(\frac{1 \text{ ft.}}{12 \text{ in.}}\right)\left(\frac{1 \text{ mi.}}{5280 \text{ ft.}}\right)$

≈ 66.2 mi.

4.) a.) (10 pts.) Use limits and the concept of continuity to determine the value of constants A and B so that the following function is continuous for all values of x. Start by drawing a "fake" graph.

$$f(x) = \begin{cases} Ax + B\ , & \text{if } x < -1 \\ 3\ , & \text{if } -1 \le x \le 2 \\ x^2 - A\ , & \text{if } x > 2 \end{cases}$$



Require two conditions :

i.) $\lim_{x \to 2^+} f(x) = 3 \to$

$\lim_{x \to 2^+} (x^2 - A) = 3 \to 4 - A = 3 \to \boxed{A = 1}$;

ii.) $\lim_{x \to -1^-} f(x) = 3 \to \lim_{x \to -1^-} (Ax + B) = 3 \to$

$-A + B = 3 \to -1 + B = 3 \to \boxed{B = 4}$

5.) (8 pts.) Determine all possible fixed points for the following recursion : $a_{n+1} = \dfrac{3a_n + 5}{a_n - 1}$

assume $\lim_{n \to \infty} a_n = L = \lim_{n \to \infty} a_{n+1} \to$

$L = \dfrac{3L + 5}{L - 1} \to L^2 - L = 3L + 5$

$\to L^2 - 4L - 5 = 0$

$\to (L - 5)(L + 1) = 0$

$\to L = 5$ or $L = -1$.

6.) (6 pts. each) Consider the function $f(x) = \dfrac{x}{3-x}$.

a.) Show algebraically that f is one-to-one.

assume $f(x_1) = f(x_2)$

$\rightarrow \dfrac{x_1}{3-x_1} = \dfrac{x_2}{3-x_2}$

$\rightarrow 3x_1 - \cancel{x_1 x_2} = 3x_2 - \cancel{x_1 x_2}$

$\rightarrow 3x_1 = 3x_2 \longrightarrow x_1 = x_2$

b.) Determine $y = f^{-1}(x)$, the inverse function for $y = f(x)$.

$Y = \dfrac{X}{3-X} \rightarrow$ (switch variables)

$\rightarrow X = \dfrac{Y}{3-Y} \rightarrow$ (solve for Y)

$\rightarrow 3X - XY = Y \rightarrow 3X = XY + Y$

$\rightarrow 3X = Y(X+1) \rightarrow Y = \dfrac{3X}{X+1}$

$\rightarrow f^{-1}(x) = \dfrac{3X}{X+1}$

7.) (6 pts. each) Find a formula for the nth term (starting with n=0) of each of the following sequences.

a.) $5, -7, 9, -11, 13, \cdots$: $4+1, -(6+1), 8+1, -(9+1), 11+1, \ldots, \boxed{}$

n : $0, 1, 2, 3, 4, \ldots, n$

$a_n = (-1)^n(2(n+2)+1) = (-1)^n(2n+5)$

b.) $5, 6, 8, 11, 15, 20, 26, \cdots$

(HINT: Use the fact that $1+2+3+4+\cdots+n = \dfrac{n(n+1)}{2}$.)

n	a_n
0	5
1	$6 = (1) + 5$
2	$8 = (1+2) + 5$
3	$11 = (1+2+3) + 5$
4	$15 = (1+2+3+4) + 5$
$\vdots$	$\vdots$
n	$a_n = (1+2+3+\cdots+n) + 5 = \dfrac{n(n+1)}{2} + 5$

8.) (7 pts.) Use the Squeeze Principle to determine the limit of the following sequence :

$a_n = \dfrac{\sin^2 n}{n^2+3}$; Begin with $-1 \le \sin n \le +1 \rightarrow$

$0 \le \sin^2 n \le 1 \rightarrow \dfrac{0}{n^2+3} \le \dfrac{\sin^2 n}{n^2+3} \le \dfrac{1}{n^2+3} \rightarrow$

$0 \le \dfrac{\sin^2 n}{n^2+3} \le \dfrac{1}{n^2+3}$; $\lim\limits_{n\to\infty} 0 = 0$ and

$\lim\limits_{n\to\infty} \dfrac{1}{n^2+3} = \dfrac{1}{\infty} = 0$ so by Squeeze

Principle $\lim\limits_{n\to\infty} \dfrac{\sin^2 n}{n^2+3} = 0$.

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) Consider the function $N = \dfrac{e^t - 5}{e^{2t} - 3e^t}$. Find all values of t for which $N = 2$.

$2 = \dfrac{e^t - 5}{e^{2t} - 3e^t} \rightarrow 2e^{2t} - 6e^t = e^t - 5 \rightarrow$

$2e^{2t} - 7e^t + 5 = 0 \rightarrow 2(e^t)^2 - 7(e^t) + 5 = 0 \rightarrow$

$(2e^t - 5)(e^t - 1) = 0 \rightarrow$

$e^t = \frac{5}{2} \rightarrow \ln e^t = \ln(\frac{5}{2}) \rightarrow \boxed{t = \ln(5/2)}$

or $e^t = 1 \rightarrow \boxed{t = 0}$