Math 17A (Fall 2006)
Kouba
Exam 3

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 7 pages, including the cover page.

6. Put units on answers where units are appropriate.

7. You will be graded on proper use of derivative notation.

8. You have until 8:50 a.m. sharp to finish the exam.

1.) (7 pts. each) Differentiate each of the following functions. DO NOT SIMPLIFY ANSWERS.

a.) $y = \sin^5(3-x)$

$$\xrightarrow{D} y' = 5\sin^4(3-x)\cdot\cos(3-x)\cdot(-1)$$

b.) $f(x) = \dfrac{2^x}{6+e^{3x}}$

$$\xrightarrow{D} f'(x) = \frac{(6+e^{3x})\cdot 2^x \ln 2 - 2^x\cdot e^{3x}\cdot 3}{(6+e^{3x})^2}$$

c.) $f(x) = (\ln x)^3 \cdot \log_4(\tan x)$

$$\xrightarrow{D} f'(x) = (\ln x)^3\cdot\frac{1}{\tan x}\cdot\sec^2 x\cdot\frac{1}{\ln 4} + 3(\ln x)^2\cdot\frac{1}{x}\cdot\log_4(\tan x)$$

d.) $f(x) = (\ln x)^x$ $\longrightarrow \ln f(x) = \ln(\ln x)^x = x\cdot\ln(\ln x) \xrightarrow{D}$

$$\frac{1}{f(x)}\cdot f'(x) = \cancel{x}\cdot\frac{1}{\ln x}\cdot\frac{1}{\cancel{x}} + (1)\cdot\ln(\ln x) \longrightarrow$$

$$f'(x) = (\ln x)^x\cdot\left\{\frac{1}{\ln x} + \ln(\ln x)\right\}$$

2.) (7 pts.) Determine the x-values for which the function $f(x) = \dfrac{x^2}{x-3}$ is increasing (↑) and decreasing (↓). DO NOT GRAPH THE FUNCTION.

$$\xrightarrow{D} f'(x) = \frac{(x-3)(2x) - x^2(1)}{(x-3)^2} = \frac{2x^2-6x-x^2}{(x-3)^2}$$

$$= \frac{x^2-6x}{(x-3)^2} = \frac{x(x-6)}{(x-3)^2} = 0$$

NO

+ 0 − | − 0 + f'

$x=0$ $x=3$ $x=6$

f is ↑ for $x<0$, $x>6$,

f is ↓ for $0<x<3$, $3<x<6$

3.) (12 pts.) For the following function f determine all absolute and relative maximum and minimum values, inflection points, and x- and y-intercepts. State clearly the x-values for which f is increasing (↑), decreasing (↓), concave up (⋃), and concave down (⋂). Neatly sketch the graph of f.

$$f(x) = x^3 - 3x^2 \quad \text{on the interval} \quad [-1, 4]$$

$\xrightarrow{D} f'(x) = 3x^2 - 6x = 3x(x-2) = 0$

$+ \quad 0 \quad - \quad 0 \quad +$ $\quad f'$

$x=-1$, $y=-4$: abs. min.
$x=0$, $y=0$: rel. max.
$x=2$, $y=-4$: abs. min.
$x=4$, $y=16$: abs. max.

$\xrightarrow{D} f''(x) = 6x - 6$
$= 6(x-1) = 0$

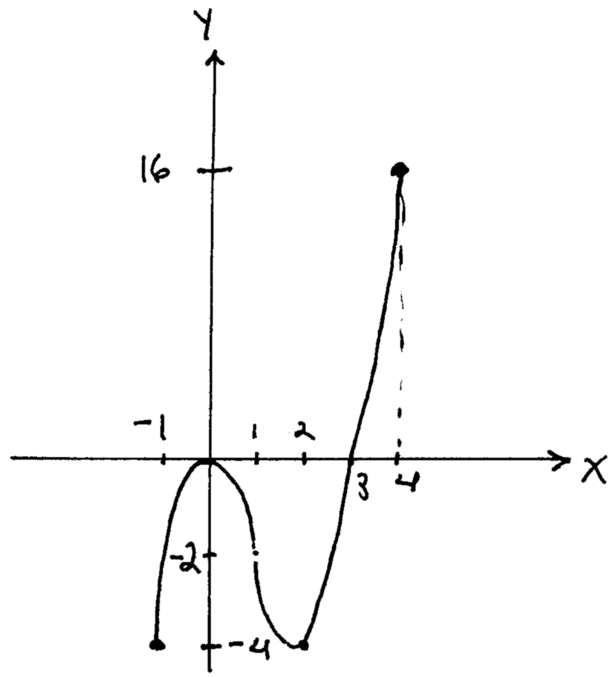
$- \quad 0 \quad +$ $\quad f''$

$x=-1 \qquad x=1 \qquad x=4$

$x=1$, $y=-2$: infl. pt.

$x=0$: $y=0$
$y=0$: $x^3 - 3x^2 = 0$
$\rightarrow x^2(x-3) = 0$
$\rightarrow x=0,\ x=3$

f is ↑ for $-1<x<0$, $2<x<4$,
f is ↓ for $0<x<2$,
f is ∪ for $1<x<4$,
f is ∩ for $-1<x<1$.



4.) (8 pts.) In 1947 earthenware jars containing what are known as the Dead Sea Scrolls were found. Analysis showed that the scrolls contained 76 % of their original carbon-14. Assuming the half-life of carbon-14 is 5730 years, estimate the age of the Dead Sea Scrolls when they were found.

assume $N = N_0 e^{kt}$, then $t = 5730$ yrs, $N = \frac{1}{2} N_0$

$\rightarrow \frac{1}{2}\cancel{N_0} = \cancel{N_0} e^{5730k} \rightarrow \ln(1/2) = \ln e^{5730k} = 5730k$

$\rightarrow k = \frac{\ln(1/2)}{5730} \rightarrow N = N_0 e^{\frac{\ln(1/2)}{5730} t}$; then

$N = 0.76 N_0 \rightarrow 0.76 \cancel{N_0} = \cancel{N_0} e^{\frac{\ln(1/2)}{5730} t} \rightarrow$

$\ln(0.76) = \ln e^{\frac{\ln(1/2)}{5730} t} = \frac{\ln(1/2)}{5730} t \rightarrow$

$$t = \frac{5730 \cdot \ln(0.76)}{\ln(1/2)} \approx 2268.7 \text{ yrs.}$$

5.) (8 pts.) Consider the function $f(x) = x - \sqrt{x}$ defined on the closed interval $[0, 4]$. Verify that f satisfies the assumptions of the Mean Value Theorem (MVT) and find all values of c guaranteed by the MVT.

$f(x) = x - \sqrt{x}$ is cont. on the closed interval $[0,4]$ since it is the difference of cont. functions; $f'(x) = 1 - \frac{1}{2\sqrt{x}}$ so f is diff. on the open interval $(0,4)$; by MVT there is a # c, $0 < c < 4$, satisfying $\frac{f(4) - f(0)}{4 - 0} = f'(c) \rightarrow$

$\frac{2-0}{4-0} = 1 - \frac{1}{2\sqrt{c}} \rightarrow \frac{1}{2\sqrt{c}} = \frac{1}{2} \rightarrow \sqrt{c} = 1 \rightarrow$

$c = 1$

$Y = 3 + X + \cos X$

6.) (8 pts.) The function $f(x) = 3 + x + \sin x$ is one-to-one and has an inverse function $y = f^{-1}(x)$. If $f(0) = 3$, then what is the value of $Df^{-1}(3)$?

$f'(X) = 1 + \cos X$ and $Df^{-1}(Y) = \dfrac{1}{f'(X)}$;

$f(0) = 3$ so $X = 0$, $Y = 3$; then

$$Df^{-1}(3) = \frac{1}{f'(0)} = \frac{1}{1 + \cos 0}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

7.) (8 pts.) Use a linearization to estimate the value of $\sqrt{68}$.

Let $f(x) = \sqrt{x}$ and $a = 64$, then $f'(x) = \dfrac{1}{2\sqrt{x}}$

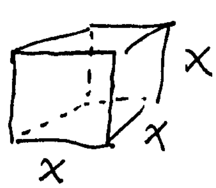
and $L(x) = f(a) + f'(a)(x - a)$

$= \sqrt{64} + \dfrac{1}{2\sqrt{64}}(x - 64) = 8 + \dfrac{1}{16}(x - 64) = 8 + \dfrac{1}{16}x - 4$

$\rightarrow L(x) = 4 + \dfrac{1}{16}x$; then

$\sqrt{68} \approx L(68) = 4 + \dfrac{1}{16}(68) = 8.25$

8.) (7 pts.) The edge x of a cube is measured with an absolute percentage error of at most 7 percent. Use a differential to estimate the maximum absolute percentage error in computing the cube's volume.



Given $\dfrac{|\Delta x|}{x} \le 7\%$, find $\dfrac{|\Delta V|}{V}$:

$V = x^3 \xrightarrow{D} V' = 3x^2$ then

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot \Delta x|}{V} = \frac{|3x^2 \cdot \Delta x|}{x^3}$$

$$= 3 \cdot \frac{|\Delta x|}{x} \le 3 \cdot 7\% = 21\%$$

9.) (7 pts.) You are to construct a closed right circular cylinder with a volume of 16π cubic feet. What radius r and height h will result in the cylinder of minimum surface area, and what is the minimum surface area ?

volume $\pi r^2 h = 16\pi$ ft.^{3} $\rightarrow$

$\boxed{h = 16/r^2}$; minimize surface area

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \cdot \frac{16}{r^2}$$

$$\rightarrow \boxed{S = 2\pi r^2 + \frac{32\pi}{r}} \xrightarrow{D} S' = 4\pi r - \frac{32\pi}{r^2}$$

$$= \frac{4\pi r^3 - 32\pi}{r^2} = \frac{4\pi(r^3 - 8)}{r^2} = 0 \rightarrow r = 2$$

$-$ 0 $+$ V'

$r = 2$ ft., $h = 4$ ft., min. $S = 24\pi$ ft.^{2}

10.) (7 pts.) There are 12 rose bushes in a garden. Each bush has 200 aphids living on it. For each additional rose bush planted in the garden, the number of aphids per bush is reduced by 10 aphids. What number of rose bushes in the garden will result in the total number of aphids being a maximum ?

Let x: # of additional rose bushes; total # of aphids T = (# of bushes)(aphids / bush) $\rightarrow$

$$T = (12 + x)(200 - 10x) \xrightarrow{D}$$

$$T' = (12 + x)(-10) + (1)(200 - 10x)$$

$$= -120 - 10x + 200 - 10x$$

$$= 80 - 20x = 0 \rightarrow x = 4$$

$+$ 0 $-$ T'

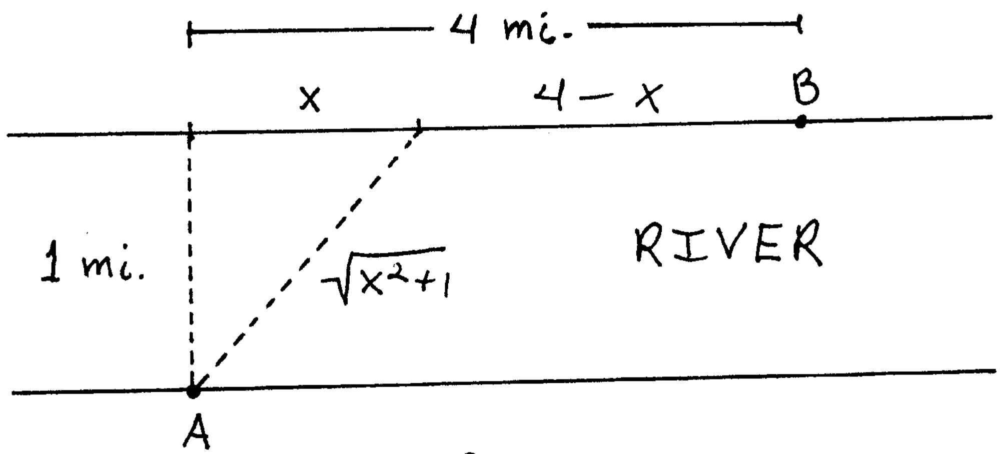
$x = 4$ extra bushes (16 total)

(160 aphids per bush)

$T = (16)(160) = 2560$ aphids

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) You are standing at point A on the edge of a river 1 mile wide. You are to get to point B, which is 4 miles from the point directly across the river from you. You can paddle a canoe in the water at a speed of 10 miles per hour and you can ride a bicycle on land at a speed of 15 miles per hour. Determine x so that the time it takes to go from point A to point B is a minimum and determine the minimum time.



$D = RT$ so $T = D/R$; minimize time

$T = T_{water} + T_{land} \rightarrow$

$$T = \frac{\sqrt{x^2+1}}{10} + \frac{4-x}{15} \xrightarrow{D}$$

$$T' = \frac{1}{10} \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x - \frac{1}{15}$$

$$= \frac{x}{10\sqrt{x^2+1}} - \frac{1}{15} = 0 \rightarrow \frac{x}{10\sqrt{x^2+1}} = \frac{1}{15} \rightarrow$$

$$15x = 10\sqrt{x^2+1} \rightarrow 225x^2 = 100(x^2+1) \rightarrow$$

$$225x^2 = 100x^2 + 100 \rightarrow 125x^2 = 100 \rightarrow$$

$$x^2 = \frac{100}{125} \rightarrow x = \sqrt{\frac{100}{125}} \approx 0.894 \text{ mi}$$

$$\overset{-\quad 0 \quad +}{\overline{\quad\quad\quad\quad}} \quad T'$$

$x = \sqrt{\frac{100}{125}}$ mi , min $T \approx 0.34$ hr.