Section 5.8: Problems

In Problems 1–40, find the general antiderivative of the given function.

1. \( f(x) = 4x^2 - x \)
2. \( f(x) = 2 - 5x^2 \)
3. \( f(x) = x^2 + 3x - 4 \)
4. \( f(x) = 3x^2 - x^4 \)
5. \( f(x) = x^4 - 3x^2 + 1 \)
6. \( f(x) = 2x^3 + x^2 - 5x \)
7. \( f(x) = 4x^3 - 2x + 3 \)
8. \( f(x) = x - 2x^2 - 3x^3 - 4x^4 \)
9. \( f(x) = 1 + \frac{1}{x} + \frac{1}{x^2} \)
10. \( f(x) = x^2 - \frac{2}{x^2} + \frac{3}{x^3} \)
11. \( f(x) = 1 - \frac{1}{x^2} \)
12. \( f(x) = x^4 - \frac{1}{x^3} \)
13. \( f(x) = \frac{1}{1 + x} \)
14. \( f(x) = \frac{x}{1 + x} \)
15. \( f(x) = 5x^4 + \frac{5}{x^4} \)
16. \( f(x) = x^7 + \frac{1}{x^7} \)
17. \( f(x) = \frac{1}{1 + 2x} \)
18. \( f(x) = \frac{1}{1 + 3x} \)
19. \( f(x) = e^{-2x} \)
20. \( f(x) = e^{x/2} + e^{-x/2} \)
21. \( f(x) = 2e^{2x} \)
22. \( f(x) = -4e^{4x} \)
23. \( f(x) = \frac{1}{e^{2x}} \)
24. \( f(x) = \frac{3}{e^{-x}} \)
25. \( f(x) = \sin(2x) \)
26. \( f(x) = \cos(3x) \)
27. \( f(x) = \sin\left(\frac{x}{3}\right) + \cos\left(\frac{x}{3}\right) \)
28. \( f(x) = \cos\left(\frac{x}{5}\right) - \sin\left(\frac{x}{5}\right) \)
29. \( f(x) = 2\sin\left(\frac{\pi}{2}x\right) - 3\cos\left(\frac{\pi}{2}x\right) \)
30. \( f(x) = -3\sin\left(\frac{\pi}{3}x\right) + 4\cos\left(-\frac{\pi}{4}x\right) \)
31. \( f(x) = \sec^2(2x) \)
32. \( f(x) = \sec^2(-4x) \)
33. \( f(x) = \sec^2\left(\frac{x}{3}\right) \)
34. \( f(x) = \sec^2\left(-\frac{x}{4}\right) \)
35. \( f(x) = \frac{\sec x + \cos x}{\cos x} \)
36. \( f(x) = \sin^2 x + \cos^2 x \)
37. \( f(x) = x^3 + 3x^5 + \sin(2x) \)
38. \( f(x) = 2e^{-3x} + \sec^2\left(-\frac{x}{2}\right) \)
39. \( f(x) = \sec^2(3x) - 1 + \frac{x^2 - 3}{x} \)
40. \( f(x) = 5e^{2x} - \sec^2(x - 3) \)

In Problems 41–46, assume that \( a \) is a positive constant. Find the general antiderivative of the given function.

41. \( f(x) = \frac{e^{a(x+1)}}{a} \)
42. \( f(x) = \sin^2(a^2x + 1) \)
43. \( f(x) = \frac{1}{ax + 3} \)
44. \( f(x) = \frac{a}{a + x} \)
45. \( f(x) = x^2 + a^2 - a^2x^2 \)
46. \( f(x) = \frac{e^{-ax} + e^{ax}}{2a} \)

In Problems 47–58, find the general solution of the differential equation.

47. \( \frac{dy}{dx} = \frac{2}{x} - x, x > 0 \)
48. \( \frac{dy}{dx} = \frac{2}{x^3} - x^3, x > 0 \)
49. \( \frac{dy}{dx} = x(1 + x), x > 0 \)
50. \( \frac{dy}{dx} = e^{-ax}, x > 0 \)
51. \( \frac{dy}{dt} = t(1 - t), t \geq 0 \)
52. \( \frac{dy}{dt} = t^2(1 - t^2), t \geq 0 \)
53. \( \frac{dy}{dt} = e^{-t/2}, t \geq 0 \)
54. \( \frac{dy}{dt} = 1 - e^{-3t}, t \geq 0 \)
55. \( \frac{dy}{ds} = \sin(\pi s), 0 \leq s \leq 1 \)
56. \( \frac{dy}{ds} = \cos(2\pi s), 0 \leq s \leq 1 \)
57. \( \frac{dy}{dx} = \sec^2\left(\frac{x}{2}\right), -1 < x < 1 \)
58. \( \frac{dy}{dx} = 1 + \sec^2\left(\frac{x}{4}\right), -1 < x < 1 \)

In Problems 59–72, solve the initial-value problem.

59. \( \frac{dy}{dx} = 3x^2, x \geq 0 \) with \( y = 1 \) when \( x = 0 \)
60. \( \frac{dy}{dx} = \frac{x^2}{3}, x \geq 0 \) with \( y = 2 \) when \( x = 0 \)
61. \( \frac{dy}{dx} = 2\sqrt{x}, x \geq 0 \) with \( y = 1 \) when \( x = 1 \)
62. \( \frac{dy}{dx} = \frac{x}{2\sqrt{x}}, x \geq 0 \) with \( y = 3 \) when \( x = 4 \)
63. \( \frac{dN}{dt} = 1, t \geq 0 \) with \( N(1) = 10 \)
64. \( \frac{dN}{dt} = \frac{t}{t + 2}, t \geq 0 \) with \( N(0) = 2 \)
65. \( \frac{dW}{dt} = e^t, t \geq 0 \) with \( W(0) = 1 \)
66. \( \frac{dW}{dt} = e^{-3t}, t \geq 0 \) with \( W(0) = 2 \)
67. \( \frac{dW}{dt} = e^{3t}, t \geq 0 \) with \( W(0) = 2/3 \)
68. \( \frac{dW}{dt} = e^{3t}, t \geq 0 \) with \( W(0) = 1 \)
69. \( \frac{dT}{dt} = \sin(\pi t), t \geq 0 \) with \( T(0) = 3 \)
70. \( \frac{dT}{dt} = \cos(\pi t), t \geq 0 \) with \( T(0) = 3 \)
71. \( \frac{dy}{dx} = \frac{e^{-x} + e^x}{2}, x \geq 0 \) with \( y = 0 \) when \( x = 0 \)
72. \( \frac{dN}{dt} = t^{-1/3}, t \geq 0 \) with \( N(0) = 60 \)

73. Suppose that the length of a certain organism at age \( x \) is given by \( L(x) \), which satisfies the differential equation

\[ \frac{dL}{dx} = e^{-0.1x}, \quad x \geq 0 \]

Find \( L(x) \) if the limiting length \( L_\infty \) is given by

\[ L_\infty = \lim_{x \to \infty} L(x) = 25 \]

How big is the organism at age \( x = 0 \)?
74. Fish are indeterminate growers; that is, their length \( L(x) \) increases with age \( x \) throughout their lifetime. If we plot the growth rate \( dL/dx \) versus age \( x \) on semilog paper, a straight line with negative slope results. Set up a differential equation that relates growth rate and age. Solve this equation under the assumption that \( L(0) = 5, L(1) = 10 \), and
\[
\lim_{x \to \infty} L(x) = 20
\]
Graph the solution \( L(x) \) as a function of \( x \).

75. An object is dropped from a height of 100 ft. Its acceleration is 32 ft/s\(^2\). When will the object hit the ground, and what will its speed be at impact?

76. Suppose that the growth rate of a population at time \( t \) undergoes seasonal fluctuations according to
\[
\frac{dN}{dt} = 3 \sin(2\pi t)
\]
where \( t \) is measured in years and \( N(t) \) denotes the size of the population at time \( t \). If \( N(0) = 10 \) (measured in thousands), find an expression for \( N(t) \). How are the seasonal fluctuations in the growth rate reflected in the population size?

77. Suppose that the amount of water contained in a plant at time \( t \) is denoted by \( V(t) \). Due to evaporation, \( V(t) \) changes over time. Suppose that the change in volume at time \( t \), measured over a 24-hour period, is proportional to \( (24 - t) \), measured in grams per hour. To offset the water loss, you water the plant at a constant rate of 4 grams of water per hour.
(a) Explain why
\[
\frac{dV}{dt} = -at(24 - t) + 4
\]
for some positive constant \( a \), describes this situation.
(b) Determine the constant \( a \) for which the net water loss over a 24-hour period is equal to 0.

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Chapter 5 Key Terms

Discuss the following definitions and concepts:
1. Global or absolute extrema
2. Local or relative extrema: local minimum and local maximum
3. The extreme-value theorem
4. Fermat's theorem
5. Mean-value theorem
6. Rolle's theorem
7. Increasing and decreasing function
8. Monotonicity and the first derivative
9. Concavity: concave up and concave down
10. Concavity and the second derivative
11. Diminishing return
12. Candidates for local extrema
13. Monotonicity and local extrema
14. The second-derivative test for local extrema
15. Inflection points
16. Inflection points and the second derivative
17. Asymptotes: horizontal, vertical, and oblique
18. Using calculus to graph functions
19. L'Hôpital's rule
20. Dynamical systems: cobwebbing
21. Stability of equilibria
22. Newton–Raphson method for finding roots
23. Antiderivative

Chapter 5 Review Problems

1. Suppose that
\[
f(x) = xe^{-x}, \quad x \geq 0
\]
(a) Show that \( f(0) = 0 \), \( f(x) > 0 \) for \( x > 0 \), and
\[
\lim_{x \to \infty} f(x) = 0
\]
(b) Find local and absolute extrema.
(c) Find inflection points.
(d) Use the foregoing information to graph \( f(x) \).

2. Suppose that
\[
f(x) = x \ln x, \quad x > 0
\]
(a) Define \( f(x) \) at \( x = 0 \) so that \( f(x) \) is continuous for all \( x \geq 0 \).
(b) Find extrema and inflection points.
(c) Graph \( f(x) \).

3. In Review Problem 17 of Chapter 2 we introduced the hyperbolic functions
\[
\sinh x = \frac{e^x - e^{-x}}{2}, \quad x \in \mathbb{R}
\]
\[
\cosh x = \frac{e^x + e^{-x}}{2}, \quad x \in \mathbb{R}
\]
\[
\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad x \in \mathbb{R}
\]
(a) Show that \( f(x) = \tanh x, \quad x \in \mathbb{R} \), is a strictly increasing function on \( \mathbb{R} \). Evaluate
\[
\lim_{x \to +\infty} \tanh x
\]
and
\[
\lim_{x \to -\infty} \tanh x
\]
(b) Use your results in (a) to explain why \( f(x) = \tanh x, \quad x \in \mathbb{R} \), is invertible, and show that its inverse function \( f^{-1}(x) = \tanh^{-1} x \) is given by
\[
f^{-1}(x) = \frac{1}{2} \ln \frac{1 + x}{1 - x}
\]
What is the domain of \( f^{-1}(x) \)?
(c) Show that
\[
\frac{d}{dx} f^{-1}(x) = \frac{1}{1 - x^2}
\]
(d) Use your result in (c) and the facts that
\[
\tanh x = \frac{\sinh x}{\cosh x}
\]
and
\[
\cosh^2 x - \sinh^2 x = 1
\]
to show that
\[
\frac{d}{dx} \tan x = \frac{1}{\cosh^2 x}
\]