Math 17 B
Kouba
Functions of Exponential Order

DEFINITION : Let $y = f(t)$ be defined for $t \geq 0$. We say f is of exponential order if there exist numbers c, $M > 0$, and $T > 0$ so that

$$\left| f(t) \right| \leq Me^{ct} \quad \text{for} \quad t > T \ .$$

QUESTION : For what functions does the Laplace transform exist ?

THEOREM : Let $y = f(t)$ be defined for $t \geq 0$. If f is

i.) continuous
and ii.) of exponential order, then

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ exists.

NOTE : If f is of exponential order, i.e., $\left| f(t) \right| \leq Me^{ct}$ for $t > T$, then

$$-Me^{ct} \leq f(t) \leq Me^{ct} \quad \longrightarrow$$

$$-Me^{cA} \leq f(A) \leq Me^{cA} \quad \longrightarrow$$

$$\frac{-Me^{cA}}{e^{sA}} \leq \frac{f(A)}{e^{sA}} \leq \frac{Me^{cA}}{e^{sA}} \quad \longrightarrow$$

$$-Me^{cA-sA} \leq \tfrac{f(A)}{e^{sA}} \leq Me^{cA-sA} \quad \longrightarrow$$

$$-Me^{(c-s)A} \leq \tfrac{f(A)}{e^{sA}} \leq Me^{(c-s)A} \quad \longrightarrow \quad \text{(Assume } s > c.\text{)}$$

$$\frac{-M}{e^{(s-c)A}} \leq \frac{f(A)}{e^{sA}} \leq \frac{M}{e^{(s-c)A}} \ .$$

Then

$$\lim_{A \to \infty} \frac{-M}{e^{(s-c)A}} = 0 = \lim_{A \to \infty} \frac{M}{e^{(s-c)A}} \ .$$

Thus, by the Squeeze Principle, $\displaystyle \lim_{A \to \infty} \frac{f(A)}{e^{sA}} = 0$.

This result will be used in the determination of additional Laplace transform formulas.