Math 17B
Kouba
Leslie Matrices – for Population Models with Discrete Breeding Seasons

We now discuss populations with discrete breeding seasons, where reproduction is limited to a particular season of the year. For example, let's consider the number of female ruby-throated hummingbirds in a population with an annual breeding season of March to July. A female usually lays one clutch of two eggs; sometimes two clutches are laid. We will assume that the average lifespan of a female hummingbird is four years. We define the age of a bird at the end of a breeding season as follows:

- age zero (0): any bird that is born during the current breeding season
- age one (1): any zero-year old bird which survives to the end of the next breeding season
- age two (2): any one-year old bird which survives to the end of the next breeding season
- age three (3): any two-year old bird which survives to the end of the next breeding season

Let \( N_x(t) \) represent the total number of female hummingbirds of age \( x \) at the end of breeding season \( t \) for \( t = 0, 1, 2, 3, 4, 5, \ldots \). We make the following assumptions about reproductive viability of female birds:

- age zero (0): not yet reproductively mature
- age one (1): will produce an average of 1.2 female offspring the next breeding season which survive
- age two (2): will produce an average of 1.5 female offspring the next breeding season which survive
- age three (3): will produce an average of 0.7 female offspring the next breeding season which survive

This can be summarized in the following equation:

\[
N_0(t + 1) = (1.2)N_1(t) + (1.5)N_2(t) + (0.7)N_3(t)
\]

We make the following assumptions about the survival rates of female birds:

- 50\% of age zero (0) females at time \( t \) survive to time \( t + 1 \);
35\% of age one (1) females at time \( t \) survive to time \( t + 1 \); 
15\% of age two (2) females at time \( t \) survive to time \( t + 1 \); 
0\% of age three (3) females at time \( t \) survive to time \( t + 1 \).

This can be summarized in the following equations:
\[
\begin{align*}
N_1(t + 1) &= (0.5)N_0(t) , \\
N_2(t + 1) &= (0.35)N_1(t) , \\
N_3(t + 1) &= (0.15)N_2(t) .
\end{align*}
\]

Let \( N(t) \) represent the total number of female hummingbirds at the end of breeding season \( t \) for \( t = 0, 1, 2, 3, 4, 5, \ldots \). Using matrix notation we get the following representations:

Let \( N(t) = \begin{pmatrix} N_0(t) \\ N_1(t) \\ N_2(t) \\ N_3(t) \end{pmatrix} \) represent the total number of female hummingbirds by age group at the end of season \( t \).

Let \( N(t + 1) = \begin{pmatrix} N_0(t + 1) \\ N_1(t + 1) \\ N_2(t + 1) \\ N_3(t + 1) \end{pmatrix} \) represent the total number of female hummingbirds by age group at the end of season \( t + 1 \).

Let \( L = \begin{pmatrix} 0 & 1.2 & 1.5 & 0.7 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.35 & 0 & 0 \\ 0 & 0 & 0.15 & 0 \end{pmatrix} \). Combining these matrices gives

\[
\begin{pmatrix} N_0(t + 1) \\ N_1(t + 1) \\ N_2(t + 1) \\ N_3(t + 1) \end{pmatrix} = \begin{pmatrix} 0 & 1.2 & 1.5 & 0.7 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.35 & 0 & 0 \\ 0 & 0 & 0.15 & 0 \end{pmatrix} \begin{pmatrix} N_0(t) \\ N_1(t) \\ N_2(t) \\ N_3(t) \end{pmatrix} , \text{ i.e.,}
\]

\[
N(t + 1) = L N(t) .
\]

The 4 by 4 matrix \( L \) is called a Leslie Matrix.