Math 17B
Kouba
Eigenvalues, Leslie Matrices, and Stable Age Distributions

WE WILL MAKE THE FOLLOWING ASSUMPTIONS ABOUT LESLIE MATRICES, L

1.) L has two distinct eigenvalues, λ_1 and λ_2, with $\lambda_1 > \lambda_2$.

2.) The larger eigenvalue $\lambda_1 > 0$ and the smaller eigenvalue $\lambda_2 < 0$.

3.) $|\lambda_2| < \lambda_1$.

4.) Eigenvalue λ_1 determines the growth rate for the population.
a.) If $0 < \lambda_1 < 1$, then the population size decreases as $t \longrightarrow \infty$.
b.) If $\lambda_1 > 1$, then the population size increases as $t \longrightarrow \infty$.

5.) Any eigenvector V_1 for eigenvalue λ_1 determines a *stable age distribution.* Assume that $V_1 = \begin{pmatrix} d \\ e \end{pmatrix}$. Being a stable age distribution means that as $t \longrightarrow \infty$ the number of zero-year olds approaches the ratio $\dfrac{d}{d+e}$ of the entire population; the number of one-year olds approaches the ratio $\dfrac{e}{d+e}$ of the entire population.

EXAMPLE : Consider the Leslie matrix $L = \begin{pmatrix} 1.5 & 2 \\ 0.08 & 0 \end{pmatrix}$. Find its eigenvalues, eigenvectors, and stable age distribution. Is the population increasing or decreasing as $t \longrightarrow \infty$?

$$L - \lambda I = \begin{pmatrix} 1.5 & 2 \\ 0.08 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1.5 & 2 \\ 0.08 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1.5 - \lambda & 2 \\ 0.08 & -\lambda \end{pmatrix} \quad \longrightarrow$$

$$det(L - \lambda I) = det \begin{pmatrix} 1.5 - \lambda & 2 \\ 0.08 & -\lambda \end{pmatrix}$$

$$= (1.5 - \lambda)(-\lambda) - (2)(0.08) = \lambda^2 - 1.5\lambda - 0.16 = 0 \quad \longrightarrow \quad \text{(Use quadratic formula.)}$$

$$\lambda = \frac{1.5 \pm \sqrt{(-1.5)^2 - (4)(1)(-0.16)}}{2} \quad \longrightarrow \quad \lambda_1 = 1.6 \text{ and } \lambda_2 = -0.1 \ .$$

Note that since $\lambda_1 = 1.6 > 1$, the population increases as $t \longrightarrow \infty$.

Now find an eigenvector for each eigenvalue by solving $(L - \lambda I)X = \mathit{\Theta}$ for X :

For $\underline{\lambda_1 = 1.6}$: $\begin{pmatrix} 1.5 - 1.6 & 2 & | & 0 \\ 0.08 & 0 - 1.6 & | & 0 \end{pmatrix} \sim \begin{pmatrix} -0.1 & 2 & | & 0 \\ 0.08 & -1.6 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -20 & | & 0 \\ 8 & -160 & | & 0 \end{pmatrix} \sim$
$\begin{pmatrix} 1 & -20 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \longrightarrow$

$x_1 - 20x_2 = 0$ so let $x_2 = t$ any number, then $x_1 = 20x_2 = 20t$ and

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 20t \\ t \end{pmatrix} = t\begin{pmatrix} 20 \\ 1 \end{pmatrix}$, so choose $V_1 = \begin{pmatrix} 20 \\ 1 \end{pmatrix}$ as an eigenvector for $\lambda_1 = 1.6$.

Note that vector $V_1 = \begin{pmatrix} 20 \\ 1 \end{pmatrix}$ is a stable age distribution. This means that as $t \longrightarrow \infty$ the number of zero-year olds is approximately $\dfrac{20}{20+1} \approx 0.9523 = 95.23\%$. In addition, the number of one-year olds is approximately $\dfrac{1}{20+1} \approx 0.0477 = 4.77\%$.

For $\underline{\lambda_2 = -0.1}$: $\begin{pmatrix} 1.5 + 0.1 & 2 & | & 0 \\ 0.08 & 0.1 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1.6 & 2 & | & 0 \\ 0.08 & 0.1 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1.25 & | & 0 \\ 8 & 10 & | & 0 \end{pmatrix} \sim$
$\begin{pmatrix} 1 & 1.25 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \longrightarrow$

$x_1 + 1.25x_2 = 0$ so let $x_2 = t$ any number, then $x_1 = -1.25x_2 = -1.25t = (-5/4)t$ and

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (-5/4)t \\ t \end{pmatrix} = (1/4)t\begin{pmatrix} -5 \\ 4 \end{pmatrix}$, so choose $V_2 = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$ as an eigenvector for $\lambda_2 = -0.1$.