Math 17B
Kouba
Examples Using Taylor Polynomials

Example A: (Taylor Error) Consider the function $f(x) = \sqrt{x+9}$ and its second degree Taylor Polynomial centered at $a=0$, $P_2(x) = 3 + \frac{1}{6}x - \frac{1}{216}x^2$. Determine the Taylor Error on the interval $0 \le x \le 1/2$.

Recall: $\left| f(x) - P_2(x) \right| = \left| R_3(x;0) \right| = \left| \frac{f'''(c)}{3!}(x-0)^3 \right|$; where c is between x and 0;

then $f(x) = (x+9)^{1/2} \xrightarrow{D} f'(x) = \frac{1}{2}(x+9)^{-1/2}$

$\xrightarrow{D} f''(x) = \frac{-1}{4}(x+9)^{-3/2} \xrightarrow{D} f'''(x) = \frac{3}{8}(x+9)^{-5/2}$.

now $\left| R_3(x;0) \right| = \left| \frac{\frac{3}{8}(c+9)^{-5/2}}{3!}(x-0)^3 \right|$

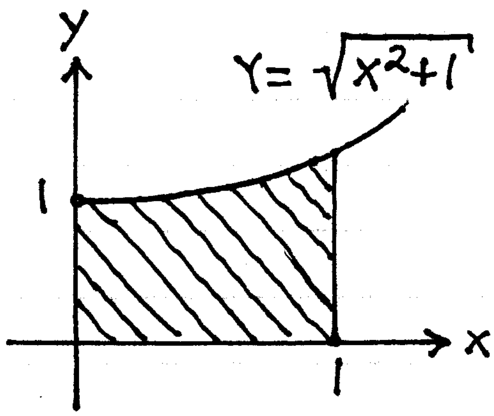
$= \frac{1}{16} \frac{|x|^3}{(c+9)^{5/2}}$, where $0 \le c \le x \le \frac{1}{2}$

$\le \frac{1}{16} \cdot \frac{|1/2|^3}{(0+9)^{5/2}}$

$= \frac{1}{16} \cdot \frac{1}{8} \cdot \frac{1}{3^5} = \frac{1}{31,104} \approx 0.000032$; thus

$\left| R_3(x;0) \right| = \left| f(x) - P_2(x) \right| \le 0.000032$
for $0 \le x \le 1/2$.

Example B: (Application of Taylor Polynomial) An Everglades habitat, whose dimensions are measured in km, is determined to lie in the region bounded by the graphs of $Y=0$, $X=0$, $X=1$, and $y=\sqrt{x^2+1}$. Use a fourth degree Taylor Polynomial centered at $a=0$ to estimate the area of the habitat.



It can be shown that the fourth degree Taylor Polynomial for $Y=\sqrt{x^2+1}$ is $P_4(x)=1+\frac{1}{2}x^2-\frac{1}{8}x^4$. Since $P_4(x)\approx f(x)$, we have

$$\text{Area} = \int_0^1 \sqrt{x^2+1}\,dx \approx \int_0^1 P_4(x)\,dx$$
$$= \int_0^1 \left(1+\tfrac{1}{2}x^2-\tfrac{1}{8}x^4\right)dx$$
$$= \left(x-\tfrac{1}{6}x^3-\tfrac{1}{40}x^5\right)\Big|_0^1$$
$$= \frac{137}{120} \approx 1.1417 \text{ km}^2 \quad ;$$

Note: Using a Taylor Error, we can show that this estimate is within about $\frac{1}{2}\%$ of the exact value.