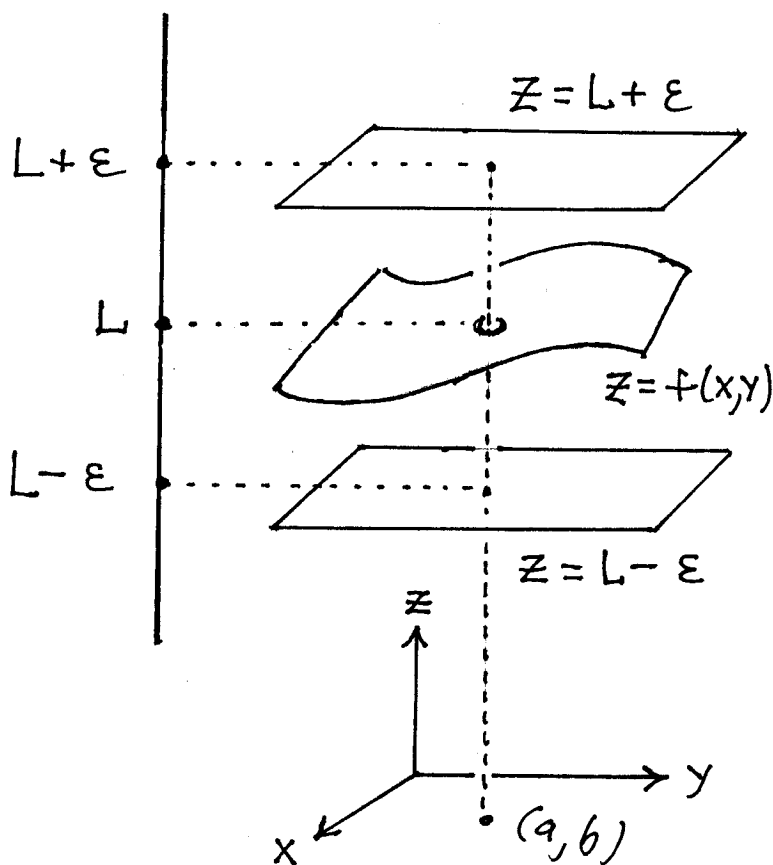
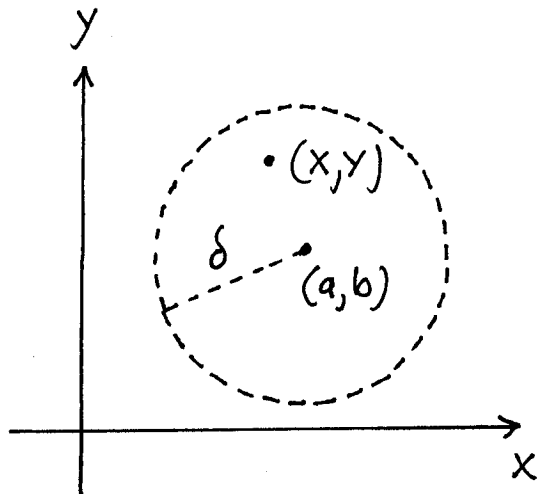


DEFINITION : Let $z = f(x, y)$ be a function. The limit $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ means that for each $\epsilon > 0$ there exist a $\delta > 0$ so that

$$\text{if } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta, \text{ then } |f(x, y) - L| < \epsilon.$$



DEFINITION : Function $z = f(x, y)$ is continuous at point (a, b) iff

- i.) $f(a, b)$ is defined (finite) ,
- ii.) $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ (finite) ,
- and iii.) $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$.

Ex: Prove that $\lim_{(x,y) \rightarrow (1,-1)} (x^2 - y) = 2$:

Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-1)^2 + (y-(-1))^2} = \sqrt{(x-1)^2 + (y+1)^2} < \delta$,
then $|(x^2 - y) - 2| < \varepsilon$. Then

$$|x^2 - y - 2| = |(x-1)^2 + 2x - 1 - (y+1) + 1 - 2|$$

$$= |(x-1)^2 + 2(x-1) - (y+1)|$$

Δ -inequality $\hookrightarrow$ $\leq |x-1|^2 + |2(x-1)| + |y+1|$

$$= (x-1)^2 + 2|x-1| + |y+1|$$

$$= (x-1)^2 + 2\sqrt{(x-1)^2} + \sqrt{(y+1)^2}$$

$$\leq (x-1)^2 + (y+1)^2 + 2\sqrt{(x-1)^2 + (y+1)^2} + \sqrt{(x-1)^2 + (y+1)^2}$$

$$= (\sqrt{(x-1)^2 + (y+1)^2})^2 + 3\sqrt{(x-1)^2 + (y+1)^2}$$

assume $\delta \leq 1$ $\leq \sqrt{(x-1)^2 + (y+1)^2} + 3\sqrt{(x-1)^2 + (y+1)^2}$

so that

$$A^2 \leq A \quad = 4\sqrt{(x-1)^2 + (y+1)^2} < \varepsilon$$

iff $\sqrt{(x-1)^2 + (y+1)^2} < \frac{1}{4} \varepsilon$.

Now choose

$\delta = \min \left\{ \frac{1}{4} \varepsilon, 1 \right\}$ and the result follows.

QED