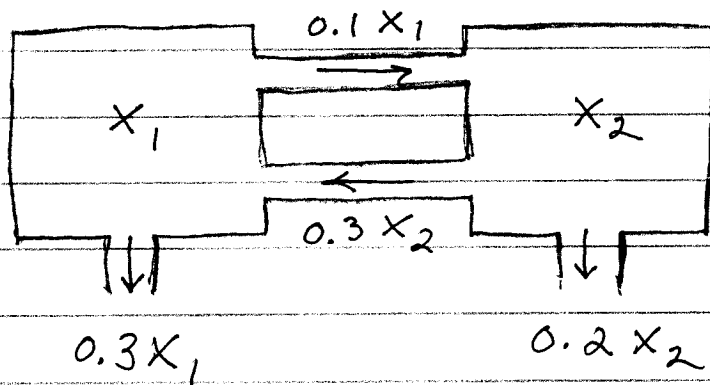


## Section 11.2

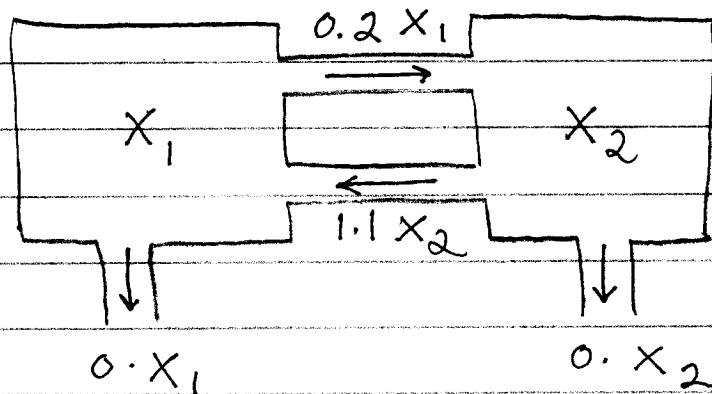
9.)  $\frac{dx_1}{dt} = -0.4x_1 + 0.3x_2$

$\frac{dx_2}{dt} = 0.1x_1 - 0.5x_2$



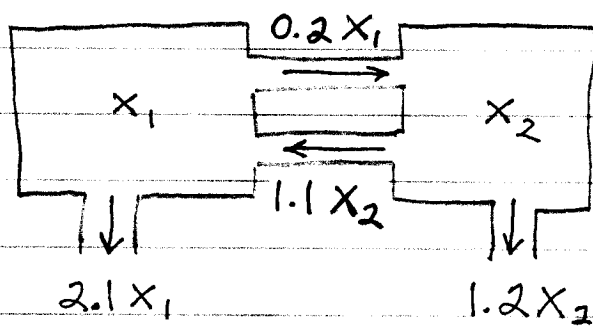
12.)  $\frac{dx_1}{dt} = -0.2x_1 + 1.1x_2$

$\frac{dx_2}{dt} = 0.2x_1 - 1.1x_2$



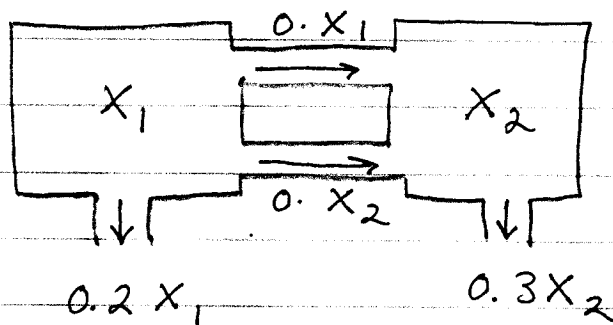
13.)  $\frac{dx_1}{dt} = -2.3x_1 + 1.1x_2$

$\frac{dx_2}{dt} = 0.2x_1 - 2.3x_2$



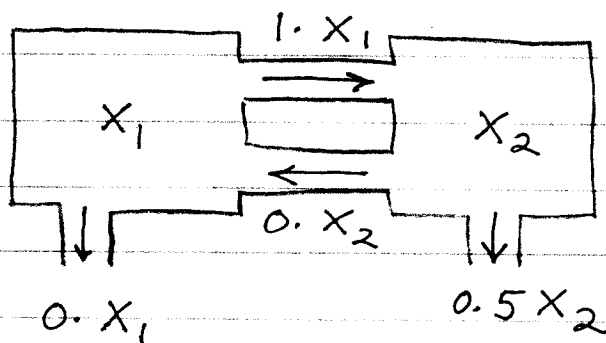
$$17.) \quad \frac{dx_1}{dt} = -0.2x_1 + 0 \cdot x_2$$

$$\frac{dx_2}{dt} = 0 \cdot x_1 + -0.3x_2$$



$$18.) \quad \frac{dx_1}{dt} = -1 \cdot x_1 + 0 \cdot x_2$$

$$\frac{dx_2}{dt} = 1 \cdot x_1 - 0.5x_2$$



$$19.) \begin{cases} \frac{dx_1}{dt} = -0.3 x_1 & , x_1(0) = 4 \text{ mg.} \\ \frac{dx_2}{dt} = +0.3 x_1 & , x_2(0) = 0 \text{ mg.} \end{cases}$$

$$\rightarrow \frac{dx_1}{dt} = -0.3 x_1 \rightarrow \int \frac{1}{x_1} dx_1 = \int -0.3 dt$$

$$\rightarrow \ln x_1 = -0.3t + c_1 \rightarrow$$

$$e^{\ln x_1} = e^{-0.3t + c_1} = e^{c_1} e^{-0.3t} \rightarrow$$

$$x_1 = c e^{-0.3t} \text{ and } x_1(0) = 4 \rightarrow$$

$$4 = c e^0 = c(1) = c \rightarrow \boxed{x_1 = 4e^{-0.3t}} ;$$

$$\text{then } \frac{dx_2}{dt} = 0.3 x_1 = (0.3) 4e^{-0.3t} \rightarrow$$

$$\frac{dx_2}{dt} = 1.2 e^{-0.3t} \rightarrow \int dx_2 = \int 1.2 e^{-0.3t} dt$$

$$\rightarrow x_2 = \frac{1.2}{-0.3} e^{-0.3t} + c \rightarrow$$

$$x_2 = -4e^{-0.3t} + c \text{ and } x_2(0) = 0 \rightarrow$$

$$0 = -4 \cdot (1) + c \rightarrow c = 4 \rightarrow \boxed{x_2 = 4 - 4e^{-0.3t}}$$

$$20.) \begin{cases} \frac{dx_1}{dt} = -a x_1 & , x_1(0) = 6 \text{ mg.} \\ \frac{dx_2}{dt} = +a x_1 & , x_2(0) = 0 \text{ mg.} \end{cases}$$

assume  $x_1(20) = \frac{1}{2}(6) = 3$  mg. ; then

$$\frac{dx_1}{dt} = -ax_1 \rightarrow \int \frac{1}{x_1} dx_1 = \int -a dt \rightarrow$$

$$\ln x_1 = -at + c_1 \rightarrow e^{\ln x_1} = e^{-at + c_1} = e^{c_1} e^{-at}$$

$$\rightarrow \underline{x_1 = ce^{-at}} \text{ and } x_1(0) = 6 \rightarrow$$

$$6 = ce^0 = c(1) = c \rightarrow \underline{x_1 = 6e^{-at}} ; \text{ and}$$

$$x_1(20) = 3 \rightarrow 3 = 6e^{-20a} \rightarrow \frac{1}{2} = e^{-20a}$$

$$\rightarrow \ln(1/2) = \ln e^{-20a} = -20a \rightarrow$$

$$\boxed{a = \frac{-1}{20} \ln(1/2)} ; \text{ then}$$

$$x_1 = 6e^{\frac{1}{20} \ln(1/2) t} = 6e^{\frac{t}{20} \ln(1/2)} = 6e^{\ln(1/2) \frac{t}{20}} \rightarrow$$

$$\boxed{x_1 = 6 \left(\frac{1}{2}\right)^{t/20}} ; \text{ then}$$

$$\frac{dx_2}{dt} = \frac{-1}{20} \ln\left(\frac{1}{2}\right) \cdot x_1 = \frac{-1}{20} \ln\left(\frac{1}{2}\right) \cdot 6 \left(\frac{1}{2}\right)^{t/20} \rightarrow$$

$$\frac{dx_2}{dt} = \frac{-3}{10} \left(\frac{1}{2}\right)^{t/20} \cdot \ln(1/2) \rightarrow$$

$$\int dx_2 = \int \frac{-3}{10} \left(\frac{1}{2}\right)^{t/20} \cdot \ln(1/2) dt \rightarrow$$

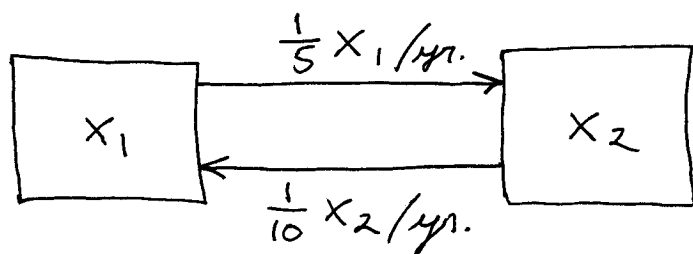
$$x_2 = \frac{-3}{10} \cdot 20 \cdot \left(\frac{1}{2}\right)^{t/20} + c \rightarrow$$

$$x_2 = -6 \left(\frac{1}{2}\right)^{t/20} + c \text{ and } x_2(0) = 0 \rightarrow$$

$$0 = -6 \left(\frac{1}{2}\right)^0 + c = -6 + c \rightarrow c = 6 \rightarrow$$

$$x_2 = 6 - 6 \left(\frac{1}{2}\right)^{t/20}$$

$$21.) \begin{cases} \frac{dx_1}{dt} = -\frac{1}{5}x_1 + \frac{1}{10}x_2 \\ a.) \frac{dx_2}{dt} = \frac{1}{5}x_1 - \frac{1}{10}x_2 \end{cases}$$



$$e.) \quad X' = \begin{bmatrix} -\frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{1}{10} \end{bmatrix} X \rightarrow$$

$$\det(A - \lambda I) = \det \begin{bmatrix} -\frac{1}{5} - \lambda & \frac{1}{10} \\ \frac{1}{5} & -\frac{1}{10} - \lambda \end{bmatrix}$$

$$= \left(-\frac{1}{5} - \lambda\right)\left(-\frac{1}{10} - \lambda\right) - \left(\frac{1}{5}\right)\left(\frac{1}{10}\right) = 0 \quad (\text{mult. by } 50) \rightarrow$$

$$5\left(-\frac{1}{5} - \lambda\right) \cdot 10\left(-\frac{1}{10} - \lambda\right) - 50\left(\frac{1}{50}\right) = 0 \rightarrow$$

$$(-1 - 5\lambda)(-1 - 10\lambda) - 1 = 0 \rightarrow$$

$$50\lambda^2 + 15\lambda + \cancel{1} = 0 \rightarrow 5\lambda(10\lambda + 3) = 0 \rightarrow$$

$$\lambda = 0, \lambda = -\frac{3}{10}; \text{ Solve } (A - \lambda I)X = 0:$$

$$\text{For } \lambda_1 = 0: \left[ \begin{array}{cc|c} -0.2 & 0.1 & 0 \\ 0.2 & -0.1 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\rightarrow x_1 - \frac{1}{2}x_2 = 0 \rightarrow \text{let } x_2 = t \text{ any } \# \rightarrow$$

$$x_1 = \frac{1}{2}t, \text{ so } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = \frac{1}{2}t \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

so eigenvector is  $V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  ;

For  $\lambda_2 = -\frac{3}{10}$  :  $\begin{bmatrix} 0.1 & 0.1 & | & 0 \\ 0.2 & 0.2 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \rightarrow$

$$x_1 + x_2 = 0 \rightarrow \text{let } x_2 = t \text{ any \#} \rightarrow$$

$$x_1 = -t, \text{ so } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so eigenvector is  $V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  ; then

general solution is

$$X = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{0t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-0.3t} \quad \text{OR}$$

$$\begin{cases} x_1 = c_1 - c_2 e^{-0.3t} & , \quad x_1(0) = 2 \\ x_2 = 2c_1 + c_2 e^{-0.3t} & , \quad x_2(0) = 18 \end{cases}$$

$$\rightarrow \begin{cases} c_1 - c_2 = 2 \\ 2c_1 + c_2 = 18 \end{cases} \rightarrow 3c_1 = 20 \rightarrow \boxed{c_1 = \frac{20}{3}}$$

$$\rightarrow c_2 = c_1 - 2 = \frac{20}{3} - \frac{6}{3} = \frac{14}{3} \rightarrow \boxed{c_2 = \frac{14}{3}} ;$$

$$\rightarrow \begin{cases} x_1 = \frac{20}{3} - \frac{14}{3} e^{-0.3t} \\ x_2 = \frac{40}{3} + \frac{14}{3} e^{-0.3t} \end{cases} ;$$

total area is  $x_1 + x_2 = 20$ , so  
fraction of area occupied by  
adult trees is

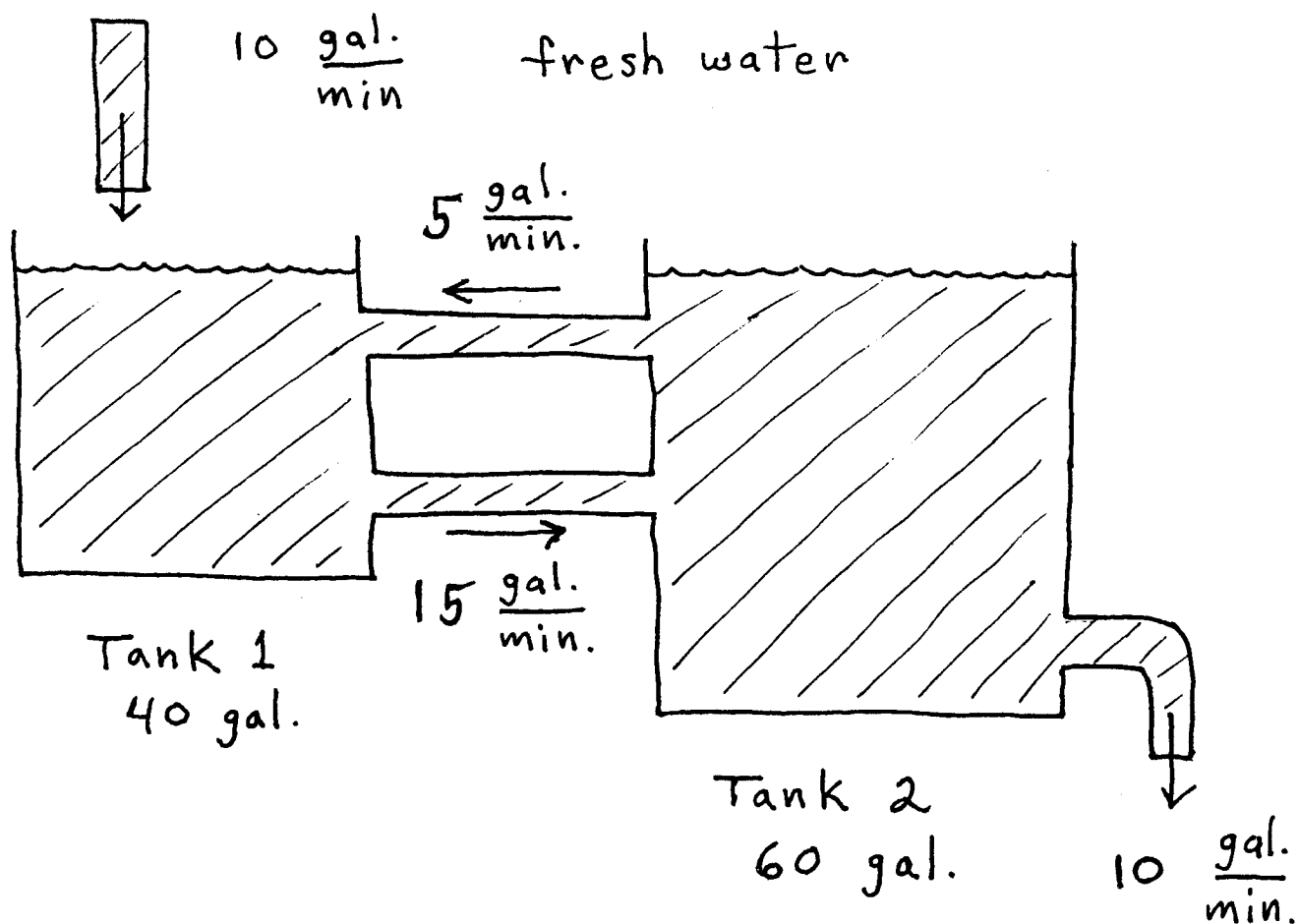
$$\begin{aligned}\frac{x_2}{20} &= \frac{1}{20} \left( \frac{40}{3} + \frac{14}{3} e^{-0.3t} \right) \\ &= \frac{2}{3} + \frac{7}{30} e^{-0.3t} ;\end{aligned}$$

$$\lim_{t \rightarrow \infty} x_1 = \lim_{t \rightarrow \infty} \left( \frac{20}{3} - \frac{14}{3} e^{-0.3t} \right) = \frac{20}{3} ;$$

$$\lim_{t \rightarrow \infty} x_2 = \lim_{t \rightarrow \infty} \left( \frac{40}{3} + \frac{14}{3} e^{-0.3t} \right) = \frac{40}{3} .$$

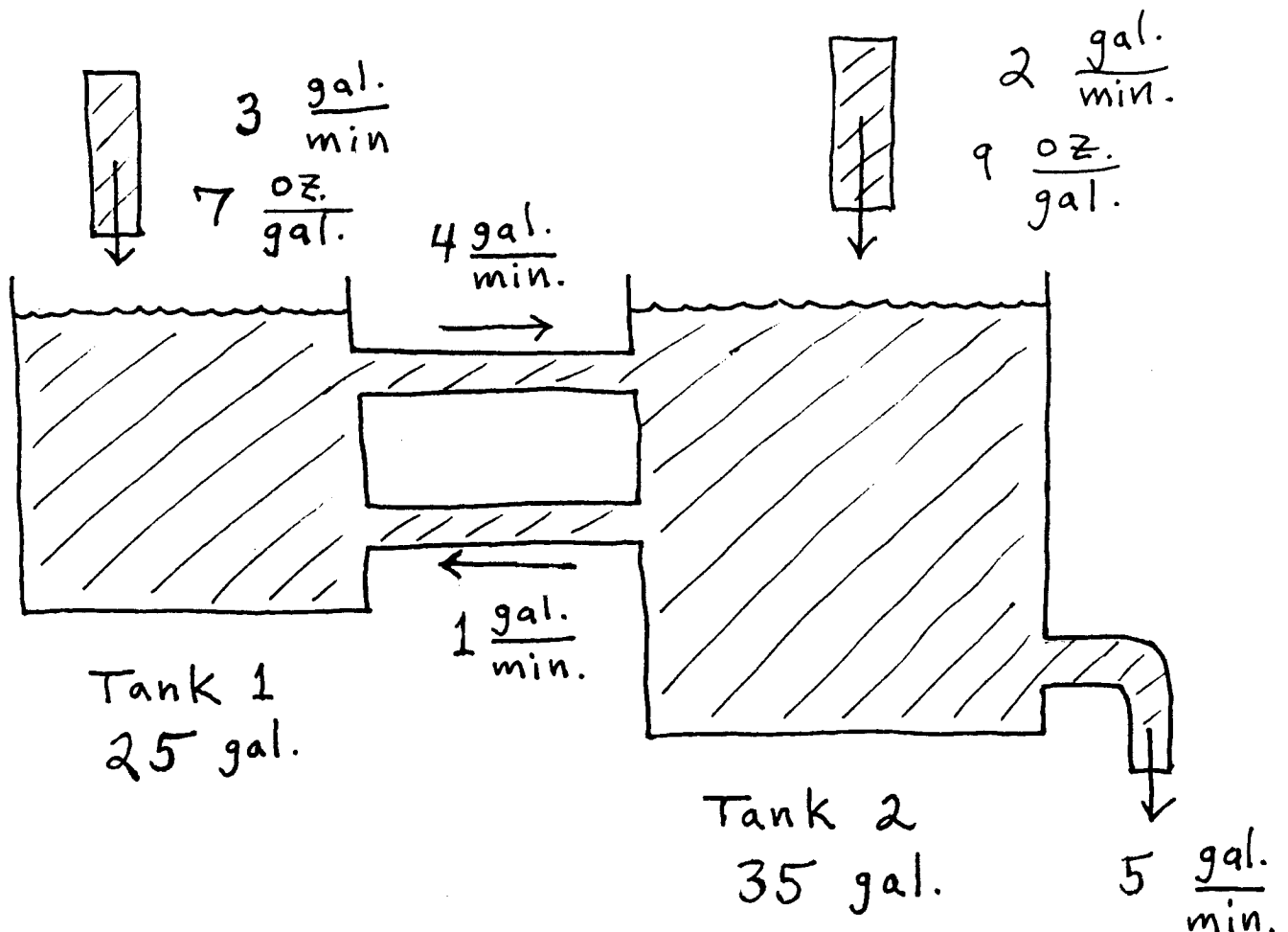
1.) Consider the two tanks containing salt water solutions and connected as shown in the diagram. Tank 1 holds 40 gallons of salt water solution. Tank 2 holds 60 gallons of salt water solution. Let  $x_1$  and  $x_2$  represent the pounds of salt in Tank 1 and Tank 2, resp., at time  $t$ . Initially, Tank 1 contains 25 pounds of salt and Tank 2 contains 15 pounds of salt. The mixture in each tank is kept uniform by stirring, and the mixtures are pumped from each tank to the other at the rates indicated in the figure. In addition, fresh water is pumped into Tank 1 at the rate of 10 gal./min.; the mixture leaves Tank 2 at 10 gal./min. Set up and solve a system of differential equations with initial conditions, which represents the amount of salt in each tank.

Let  $x_1$ : lbs. of salt in Tank 1 at time  $t$   
 $x_2$ : lbs. of salt in Tank 2 at time  $t$   
 $t$ : minutes



2.) Consider the two tanks containing milk and Hershey's chocolate syrup mixtures and connected as shown in the diagram. Tank 1 holds 25 gallons of mixture. Tank 2 holds 35 gallons of mixture. Let  $x_1$  and  $x_2$  represent the ounces of Hershey's syrup in Tank 1 and Tank 2, resp., at time  $t$ . Initially, Tank 1 contains 50 ounces of syrup and Tank 2 contains 30 ounces of syrup. The mixture in each tank is kept uniform by stirring, and the mixtures are pumped from each tank to the other at the rates indicated in the figure. In addition, a mixture containing 7 ounces of syrup per gallon is pumped into Tank 1 at 3 gal./min.; a mixture containing 9 ounces of syrup per gallon is pumped into Tank 2 at 2 gal./min.; the mixture leaves Tank 2 at 5 gal./min. SET UP, BUT DO NOT SOLVE, a system of differential equations with initial conditions, which represents the amount of Hershey's syrup in each tank.

Let  $x_1$  : oz's. of syrup in Tank 1 at time  $t$   
 $x_2$  : oz's. of syrup in Tank 2 at time  $t$   
 $t$  : minutes



# Worksheet 4

$$\begin{aligned}
 1.) \quad \frac{dx_1}{dt} &= (\text{Rate In}) - (\text{Rate Out}) \quad \left(\frac{\text{lbs.}}{\text{min.}}\right) \\
 &= \left(\frac{0 \text{ lbs.}}{\text{gal.}}\right)\left(\frac{10 \text{ gal.}}{\text{min.}}\right) + \left(\frac{x_2 \text{ lbs.}}{60 \text{ gal.}}\right)\left(\frac{5 \text{ gal.}}{\text{min.}}\right) \\
 &\quad - \left(\frac{x_1 \text{ lbs.}}{40 \text{ gal.}}\right)\left(\frac{15 \text{ gal.}}{\text{min.}}\right) ;
 \end{aligned}$$

$$\begin{aligned}
 \frac{dx_2}{dt} &= (\text{Rate In}) - (\text{Rate Out}) \quad \left(\frac{\text{lbs.}}{\text{min.}}\right) \\
 &= \left(\frac{x_1 \text{ lbs.}}{40 \text{ gal.}}\right)\left(\frac{15 \text{ gal.}}{\text{min.}}\right) \\
 &\quad - \left(\frac{x_2 \text{ lbs.}}{60 \text{ gal.}}\right)\left(\frac{5 \text{ gal.}}{\text{min.}}\right) - \left(\frac{x_2 \text{ lbs.}}{60 \text{ gal.}}\right)\left(\frac{10 \text{ gal.}}{\text{min.}}\right) ;
 \end{aligned}$$

$$\rightarrow \begin{cases} \frac{dx_1}{dt} = -\frac{3}{8}x_1 + \frac{1}{12}x_2 & , \quad x_1(0) = 25 \\ \frac{dx_2}{dt} = \frac{3}{8}x_1 - \frac{1}{4}x_2 & , \quad x_2(0) = 15 \end{cases}$$

$$\rightarrow X' = \begin{bmatrix} -3/8 & 1/12 \\ 3/8 & -1/4 \end{bmatrix} X \rightarrow$$

$$\det(A - \lambda I) = \det \begin{bmatrix} -3/8 - \lambda & 1/12 \\ 3/8 & -1/4 - \lambda \end{bmatrix}$$

$$= \left(-\frac{3}{8} - \lambda\right)\left(-\frac{1}{4} - \lambda\right) - \left(\frac{3}{8}\right)\left(\frac{1}{12}\right) = 0 \quad (\text{mult. by } 32) \rightarrow$$

$$8\left(-\frac{3}{8} - \lambda\right) \cdot 4\left(-\frac{1}{4} - \lambda\right) - 32\left(\frac{1}{32}\right) = 0 \rightarrow$$

$$(-3 - 8\lambda)(-1 - 4\lambda) - 1 = 0 \rightarrow$$

$$32\lambda^2 + 20\lambda + 3 - 1 = 0 \rightarrow$$

$$\boxed{16\lambda^2 + 10\lambda + 1 = 0} ; \text{ then}$$

$$\lambda = \frac{-10 \pm \sqrt{(10)^2 - 4(16)(1)}}{2(16)} = \frac{-10 \pm \sqrt{36}}{32} \rightarrow$$

$$\lambda = \frac{-10+6}{32} = \frac{-4}{32} = \frac{-1}{8} = -0.125 \quad \text{or}$$

$$\lambda = \frac{-10-6}{32} = -\frac{1}{2} = -0.5 ; \text{ Solve } (A - \lambda I)X = 0 :$$

For  $\lambda_1 = -0.125$  :

$$\begin{bmatrix} -0.25 & 0.08333 & | & 0 \\ 0.375 & -0.125 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -0.33332 & | & 0 \\ 1 & -0.33333 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -0.33332 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \rightarrow x_1 - 0.33332 x_2 = 0 \rightarrow$$

let  $x_2 = t$  any #  $\rightarrow x_1 = 0.33332t$ , so

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.33332t \\ t \end{bmatrix} = t \begin{bmatrix} 0.33332 \\ 1 \end{bmatrix}, \text{ so}$$

eigenvector is  $V_1 = \begin{bmatrix} 0.33332 \\ 1 \end{bmatrix} ;$

For  $\lambda_2 = -0.5$  :

$$\begin{bmatrix} 0.125 & 0.08333 & | & 0 \\ 0.375 & 0.25 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.66664 & | & 0 \\ 1 & 0.66666 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0.66664 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \rightarrow x_1 + 0.66664 x_2 = 0 \rightarrow$$

let  $x_2 = t$  any #  $\rightarrow x_1 = -0.66664t \rightarrow$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -0.66664t \\ t \end{bmatrix} = t \begin{bmatrix} -0.66664 \\ 1 \end{bmatrix},$$

so eigenvector is  
general solution

$$V_2 = \begin{bmatrix} -0.66664 \\ 1 \end{bmatrix};$$
$$\text{is } X = c_1 \begin{bmatrix} 0.33332 \\ 1 \end{bmatrix} e^{-0.125t} + c_2 \begin{bmatrix} -0.66664 \\ 1 \end{bmatrix} e^{-0.5t}$$

OR

$$\begin{cases} x_1 = 0.33332 c_1 e^{-0.125t} - 0.66664 c_2 e^{-0.5t} \\ x_2 = c_1 e^{-0.125t} + c_2 e^{-0.5t} \end{cases};$$

then

$$\begin{cases} x_1(0) = 0.33332 c_1 - 0.66664 c_2 = 25 \\ x_2(0) = c_1 + c_2 = 15 \end{cases}$$

$$\rightarrow c_1 = 15 - c_2 \rightarrow \text{(sub)}$$

$$0.33332 (15 - c_2) - 0.66664 c_2 = 25 \rightarrow$$

$$4.9998 - 0.33332 c_2 - 0.66664 c_2 = 25 \rightarrow$$

$$-0.99996 c_2 = 20.0002 \rightarrow c_2 = -20$$

$$\rightarrow c_1 = 35 \rightarrow$$

$$\begin{cases} x_1 = 11.6662 e^{-0.125t} + 13.3328 e^{-0.5t} \\ x_2 = 35 e^{-0.125t} - 20 e^{-0.5t} \end{cases}$$

$$\begin{aligned}
 2.) \quad \frac{dx_1}{dt} &= (\text{Rate In}) - (\text{Rate Out}) \quad \left( \frac{\text{oz.}}{\text{min.}} \right) \\
 &= \left( \frac{7 \text{ oz.}}{\text{gal.}} \right) \left( \frac{3 \text{ gal.}}{\text{min.}} \right) + \left( \frac{x_2 \text{ oz.}}{35 \text{ gal.}} \right) \left( \frac{1 \text{ gal.}}{\text{min.}} \right) \\
 &\quad - \left( \frac{x_1 \text{ oz.}}{25 \text{ gal.}} \right) \left( \frac{4 \text{ gal.}}{\text{min.}} \right) ;
 \end{aligned}$$

$$\begin{aligned}
 \frac{dx_2}{dt} &= (\text{Rate In}) - (\text{Rate Out}) \quad \left( \frac{\text{oz.}}{\text{min.}} \right) \\
 &= \left( \frac{9 \text{ oz.}}{\text{gal.}} \right) \left( \frac{2 \text{ gal.}}{\text{min.}} \right) + \left( \frac{x_1 \text{ oz.}}{25 \text{ gal.}} \right) \left( \frac{4 \text{ gal.}}{\text{min.}} \right) \\
 &\quad - \left( \frac{x_2 \text{ oz.}}{35 \text{ gal.}} \right) \left( \frac{1 \text{ gal.}}{\text{min.}} \right) - \left( \frac{x_2 \text{ oz.}}{35 \text{ gal.}} \right) \left( \frac{5 \text{ gal.}}{\text{min.}} \right) ;
 \end{aligned}$$

$$\begin{cases} \frac{dx_1}{dt} = -\frac{4}{25} x_1 + \frac{1}{35} x_2 + 21, & x_1(0) = 50 \\ \frac{dx_2}{dt} = \frac{4}{25} x_1 - \frac{6}{35} x_2 + 18, & x_2(0) = 30 \end{cases}$$