Section 9.3

63.) $A = \begin{bmatrix} 2 & 4 \\ -2 & -3 \end{bmatrix}$ so $\operatorname{tr} A = 2 + (-3) = -1 < 0$ and $\det A = (2)(-3) - (4)(-2) = 2 > 0 \longrightarrow$ real parts of both eigenvalues are negative

65.) $A = \begin{bmatrix} 4 & 4 \\ -4 & -3 \end{bmatrix}$ so $\operatorname{tr} A = 4 + (-3) = +1 > 0 \rightarrow$ real parts of both eigenvalues are <u>not</u> negative

67.) $A = \begin{bmatrix} 2 & -5 \\ 2 & -3 \end{bmatrix}$ so $\operatorname{tr} A = 2 + (-3) < 0$ and $\det A = (2)(-3) - (-5)(2) = 4 > 0 \longrightarrow$ real parts of both eigenvalues are negative

68.) $A = \begin{bmatrix} -2 & 5 \\ 2 & 3 \end{bmatrix}$ so $\operatorname{tr} A = -2 + (3) = +1 > 0$ and real parts of both eigenvalues are <u>not</u> negative

Section 11.3

1.) $\begin{cases} \dfrac{dx_1}{dt} = x_1 - 2x_2 + x_1 x_2 \\ \dfrac{dx_2}{dt} = -x_1 + x_2 \end{cases}$

$$Df(x_1, x_2) = \begin{bmatrix} 1+x_2 & -2+x_1 \\ -1 & 1 \end{bmatrix} \rightarrow$$

$$Df(0,0) = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = A \rightarrow$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -2 \\ -1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)(1-\lambda) - (-2)(-1) = \lambda^2 - 2\lambda + 1 - 2$$

$$= \lambda^2 - 2\lambda - 1 = 0 \longrightarrow$$

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2} \rightarrow \lambda_1 = 1 + \sqrt{2} > 0,$$

$\lambda_2 = 1 - \sqrt{2} < 0 \longrightarrow$ $(0,0)$ is unstable equilibrium (saddle)

2.) $\begin{cases} \dfrac{dx_1}{dt} = -x_1 - x_2 + x_1^2 \\ \dfrac{dx_2}{dt} = x_2 - x_1^2 \end{cases}$

$$Df(x_1, x_2) = \begin{bmatrix} -1+2x_1 & -1 \\ -2x_1 & 1 \end{bmatrix} \longrightarrow$$

$$Df(0,0) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = A \longrightarrow$$

$$\det(A-\lambda I) = \det\begin{bmatrix} -1-\lambda & -1 \\ 0 & 1-\lambda \end{bmatrix}$$

$= (-1-\lambda)(1-\lambda) - (0)(-1)$
$= \lambda^2 - 1 = (\lambda-1)(\lambda+1) = 0 \rightarrow \lambda_1 = 1,\ \lambda_2 = -1 \rightarrow$
$(0,0)$ is unstable equilibrium (saddle)

3.) $\begin{cases} \dfrac{dx_1}{dt} = x_1 + x_1^2 - 2x_1x_2 + x_2 \\ \dfrac{dx_2}{dt} = x_1 \end{cases}$

$$Df(x_1,x_2) = \begin{bmatrix} 1+2x_1-2x_2 & -2x_1+1 \\ 1 & 0 \end{bmatrix} \rightarrow$$

$$Df(0,0) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = A \rightarrow$$

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix}$$

$= \lambda^2 - \lambda - 1 = 0 \rightarrow$

$$\lambda = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2} \rightarrow$$

$\lambda_1 = \frac{1}{2}(1+\sqrt{5}) > 0,\ \lambda_2 = \frac{1}{2}(1-\sqrt{5}) < 0 \rightarrow$
$(0,0)$ is unstable equilibrium (saddle)

7.) $\begin{cases} \dfrac{dx_1}{dt} = -x_1 + 2x_1 - 2x_1^2 = x_1 - 2x_1^2 \\ \dfrac{dx_2}{dt} = -x_2 + 5x_2 - 5x_1x_2 - 5x_2^2 = 4x_2 - 5x_1x_2 - 5x_2^2 \end{cases}$

$\rightarrow x_1(1-2x_1) = 0 \rightarrow \underline{x_1 = 0}$ OR $\underline{x_1 = 1/2}$; and
$x_2(4 - 5x_1 - 5x_2) = 0 \rightarrow \underline{x_2 = 0}$ OR $4 - 5x_1 - 5x_2 = 0$;

If $x_1 = 0$ and $x_2 = 0 \rightarrow \boxed{(0,0)}$ is equilibrium;
If $x_1 = 0$, then $4 - 5(0) - 5x_2 = 0 \rightarrow 4 = 5x_2 \rightarrow$
$x_2 = 4/5$ so $\boxed{(0, 4/5)}$ is equilibrium;
If $x_1 = 1/2$ and $x_2 = 0 \rightarrow \boxed{(1/2, 0)}$ is equilibrium;
If $x_1 = 1/2$, then $4 - 5(1/2) - 5x_2 = 0 \rightarrow \frac{3}{2} = 5x_2 \rightarrow$
$x_2 = \frac{3}{10} \rightarrow \boxed{\left(\frac{1}{2}, \frac{3}{10}\right)}$ is equilibrium:

$$Df(x_1, x_2) = \begin{bmatrix} 1 - 4x_1 & 0 \\ -5x_2 & 4 - 5x_1 - 10x_2 \end{bmatrix}$$ then

a.) $Df(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = A \rightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) = 0 \rightarrow$

$\lambda_1 = 1,\ \lambda_2 = 4 \rightarrow (0,0)$ is unstable equilibrium (source)

b.) $Df(0, 4/5) = \begin{bmatrix} 1 & 0 \\ -4 & -4 \end{bmatrix} = A \rightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 \\ -4 & -4-\lambda \end{bmatrix} = (1-\lambda)(-4-\lambda) = 0$

$\rightarrow \lambda_1 = 1,\ \lambda_2 = -4 \rightarrow (0, \frac{4}{5})$ is unstable equilibrium (saddle)

c.) $Df\left(\frac{1}{2}, 0\right) = \begin{bmatrix} -1 & 0 \\ 0 & 3/2 \end{bmatrix} = A \rightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 0 \\ 0 & \frac{3}{2}-\lambda \end{bmatrix}$

$= (-1-\lambda)\left(\frac{3}{2}-\lambda\right) = 0 \rightarrow \lambda_1 = -1,\ \lambda_2 = 3/2 \rightarrow$

$\left(\frac{1}{2}, 0\right)$ is unstable equilibrium (saddle)

d.) $Df\left(1/2, 3/10\right) = \begin{bmatrix} -1 & 0 \\ -\frac{3}{2} & -3/2 \end{bmatrix} = A \rightarrow$

$\det(A-\lambda I) = \det \begin{bmatrix} -1-\lambda & 0 \\ -3/2 & -3/2-\lambda \end{bmatrix}$

$= (-1-\lambda)(-3/2-\lambda) = 0 \rightarrow \lambda_1 = -1,\ \lambda_2 = -3/2 \rightarrow$

$\left(\frac{1}{2}, \frac{3}{10}\right)$ is stable equilibrium (sink)

9.) $\begin{cases} \frac{dx_1}{dt} = 4x_1 - 4x_1^2 - 2x_1x_2 \\ \frac{dx_2}{dt} = 2x_2 - x_2^2 - x_2 = x_2 - x_2^2 \end{cases}$

$\rightarrow x_2(1-x_2) = 0 \rightarrow \underline{x_2 = 0}$ OR $\underline{x_2 = 1}$; and

$2x_1(2 - 2x_1 - x_2) = 0 \rightarrow \underline{x_1 = 0}$ OR $2 - 2x_1 - x_2 = 0$;

If $x_2 = 0$ and $x_1 = 0 \rightarrow \boxed{(0,0)}$ is equilibrium;

If $x_2 = 0$, then $2 - 2x_1 - (0) = 0 \rightarrow x_1 = 1 \rightarrow$

$\boxed{(1,0)}$ is equilibrium ;

If $x_2 = 1$ and $x_1 = 0 \rightarrow \boxed{(0,1)}$ is equilibrium;

If $x_2 = 1$, then $2 - 2x_1 - 1 = 0 \rightarrow x_1 = 1/2 \rightarrow$

$\boxed{(1/2, 1)}$ is equilibrium ;

$Df(x_1, x_2) = \begin{bmatrix} 4 - 8x_1 - 2x_2 & -2x_1 \\ 0 & 1 - 2x_2 \end{bmatrix}$ then

a.) $Df(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = A \rightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}$

$= (4-\lambda)(1-\lambda) = 0 \rightarrow \lambda_1 = 4,\ \lambda_2 = 1 \rightarrow$
$(0,0)$ is unstable equilibrium (source)

b.) $Df(1,0) = \begin{bmatrix} -4 & -2 \\ 0 & 1 \end{bmatrix} = A \rightarrow$
$\det(A - \lambda I) = \det\begin{bmatrix} -4-\lambda & -2 \\ 0 & 1-\lambda \end{bmatrix}$
$= (-4-\lambda)(1-\lambda) = 0 \rightarrow \lambda_1 = -4,\ \lambda_2 = 1 \rightarrow$
$(1,0)$ is unstable equilibrium (saddle)

c.) $Df(0,1) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = A \rightarrow$
$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda) = 0 \rightarrow$
$\lambda_1 = 2,\ \lambda_2 = -1 \rightarrow$ $(0,1)$ is unstable equilibrium (saddle)

d.) $Df(\frac{1}{2},1) = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix} = A \rightarrow$
$\det(A-\lambda I) = \det\begin{bmatrix} -2-\lambda & -1 \\ 0 & -1-\lambda \end{bmatrix}$
$= (-2-\lambda)(-1-\lambda) = 0 \rightarrow \lambda_1 = -2,\ \lambda_2 = -1 \rightarrow$
$(1/2, 1)$ is stable equilibrium (sink)

11.) $\begin{cases} \dfrac{dx_1}{dt} = x_1 - x_2 \\ \dfrac{dx_2}{dt} = x_1 x_2 - x_2 \end{cases}$

$\rightarrow$ $x_2(x_1 - 1) = 0 \rightarrow \underline{x_2 = 0}$ OR $\underline{x_1 = 1}$; and
$x_1 - x_2 = 0 \rightarrow \underline{x_1 = x_2}$; if $x_2 = 0$, ~~then~~

$x_1 = 0 \rightarrow \boxed{(0,0)}$ is equilibrium ;
If $x_1 = 1$, then $x_2 = 1 \rightarrow \boxed{(1,1)}$ is equilibrium :

$$Df(x_1, x_2) = \begin{bmatrix} 1 & -1 \\ x_2 & x_1 - 1 \end{bmatrix} \text{ then}$$

a.) $Df(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = A \rightarrow$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -1 \\ 0 & -1-\lambda \end{bmatrix}$$

$= (1-\lambda)(-1-\lambda) = 0 \rightarrow \lambda_1 = 1,\ \lambda_2 = -1 \rightarrow$
$(0,0)$ is unstable equilibrium (saddle)

b.) $Df(1,1) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = A \rightarrow$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 0-\lambda \end{bmatrix}$$

$= (1-\lambda)(-\lambda) - (1)(-1) = \lambda^2 - \lambda + 1 = 0 \rightarrow$

$$\lambda = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{(3)(-1)}}{2} \rightarrow$$

$\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$, $\lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2} i$ and $a = \frac{1}{2} > 0$

$\rightarrow$ $(1,1)$ is unstable equilibrium (unstable spiral)

5.) $\begin{cases} \dfrac{dx_1}{dt} = 10x_1 - 2x_1^2 - x_1x_2 = x_1(10-2x_1-x_2) \\ \dfrac{dx_2}{dt} = 10x_2 - x_1x_2 - 2x_2^2 = x_2(10-x_1-2x_2) \end{cases}$

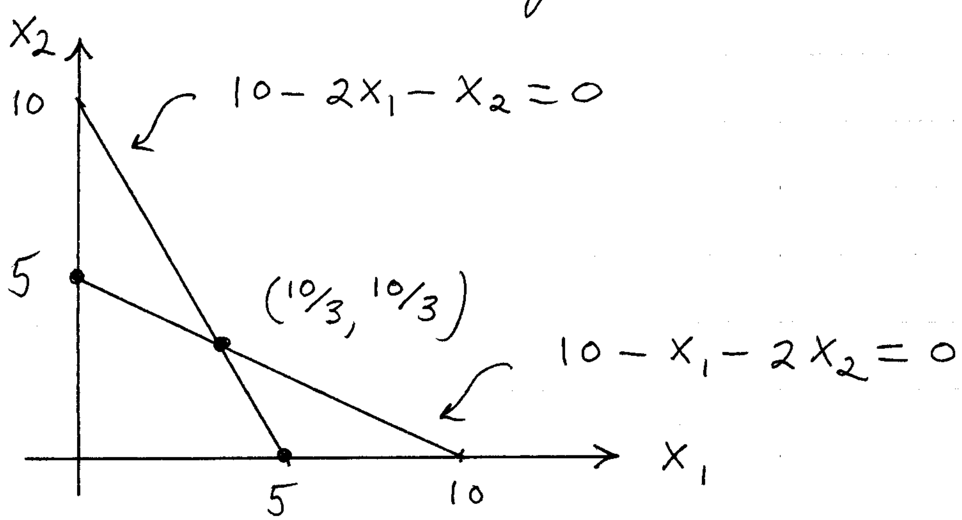
a.) $\dfrac{dx_1}{dt} = x_1(10-2x_1-x_2) = 0 \rightarrow$

$x_1 = 0$ or $10 - 2x_1 - x_2 = 0$ (lines);

$\dfrac{dx_2}{dt} = x_2(10 - x_1 - 2x_2) = 0 \longrightarrow$

$x_2 = 0$ or $10 - x_1 - 2x_2 = 0$ (lines)

all four lines are zero isoclines



b.) If $(x_1, x_2) = \left(\frac{10}{3}, \frac{10}{3}\right)$, then

$10 - 2x_1 - x_2 = 10 - 2\left(\frac{10}{3}\right) - \left(\frac{10}{3}\right) = 0$ and

$10 - x_1 - 2x_2 = 10 - \left(\frac{10}{3}\right) - 2\left(\frac{10}{3}\right) = 0$, so

$\left(\frac{10}{3}, \frac{10}{3}\right)$ is an equilibrium:

$$Df(x_1, x_2) = \begin{bmatrix} 10-4x_1-x_2 & -x_1 \\ -x_2 & 10-x_1-4x_2 \end{bmatrix} \rightarrow$$

$$Df\left(\frac{10}{3}, \frac{10}{3}\right) = \begin{bmatrix} 10 - \frac{40}{3} - \frac{10}{3} & -\frac{10}{3} \\ -\frac{10}{3} & 10 - \frac{10}{3} - \frac{40}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{20}{3} & -\frac{10}{3} \\ -\frac{10}{3} & -\frac{20}{3} \end{bmatrix} = A \longrightarrow$$

$$\det(A - \lambda I) = \det \begin{bmatrix} -\frac{20}{3} - \lambda & -\frac{10}{3} \\ -\frac{10}{3} & -\frac{20}{3} - \lambda \end{bmatrix}$$

$$= \left(-\frac{20}{3} - \lambda\right)\left(-\frac{20}{3} - \lambda\right) - \left(-\frac{10}{3}\right)\left(-\frac{10}{3}\right)$$

$$= \lambda^2 + \frac{40}{3}\lambda + \frac{400}{9} - \frac{100}{9} = \lambda^2 + \frac{40}{3}\lambda + \frac{300}{9} = 0 \rightarrow$$

$$9\lambda^2 + 120\lambda + 300 = 0 \rightarrow$$

$$3(3\lambda^2 + 40\lambda + 100) = 0 \longrightarrow$$

$$\lambda = \frac{-40 \pm \sqrt{(40)^2 - 4(3)(100)}}{2(3)} = \frac{-40 \pm \sqrt{400}}{6}$$

$$= \frac{-40 \pm 20}{6} = \frac{-20}{6} \text{ or } \frac{-60}{6} \longrightarrow$$

$\lambda_1 = -\frac{10}{3}$, $\lambda_2 = -10 \longrightarrow \left(\frac{10}{3}, \frac{10}{3}\right)$ is stable equilibrium (sink)

17.) <u>For x_1 arrow</u> :

f_1 goes from $(-)$ to $(+)$

so f_1 is $\uparrow$ and $\frac{\partial f_1}{\partial x_1}$ is $(+)$;

f_2 goes from $(-)$ to $(+)$

so f_2 is $\uparrow$ and $\frac{\partial f_2}{\partial x_1}$ is $(+)$;

<u>For x_2 arrow</u> :

f_1 goes from $(-)$ to $(+)$

so f_1 is $\uparrow$ and $\frac{\partial f_1}{\partial x_2}$ is $(+)$;

f_2 goes from $(-)$ to $(+)$

so f_2 is $\uparrow$ and $\frac{\partial f_2}{\partial x_2}$ is $(+)$;

the signs for entries of Jacobi Matrix are :

$$Df(a,b) = \begin{bmatrix} + & + \\ + & + \end{bmatrix} = A \text{ ; then}$$

tr A is $(+)$, so real parts of both eigenvalues are <u>not</u> negative, so (a,b) is <u>unstable</u> <u>equilibrium</u>.

19.) <u>For x_1 arrow</u>:

f_1 goes from $(+)$ to $(-)$

so $f_1 \downarrow$ so $\frac{\partial f_1}{\partial x_1}$ is $(-)$;

f_2 goes from $(+)$ to $(-)$

so $f_2 \downarrow$ so $\frac{\partial f_2}{\partial x_1}$ is $(-)$;

<u>For x_2 arrow</u>:

f_1 goes from $(-)$ to $(+)$

so $f_1 \uparrow$ so $\frac{\partial f_1}{\partial x_2}$ is $(+)$;

f_2 goes from $(+)$ to $(-)$

so $f_2 \downarrow$ so $\frac{\partial f_2}{\partial x_2}$ is $(-)$;

the signs for entries of Jacobi Matrix are:

$$Df(a,b) = \begin{bmatrix} - & + \\ - & - \end{bmatrix} = A, \text{ then}$$

tr A is $(-)$ and det A is $(+)$, so real parts of both eigenvalues are negative $\longrightarrow$ (a,b) is stable equilibrium.

21.) <u>For x_1 arrow</u>:

f_1 goes from $(-)$ to $(+)$

so $f_1 \uparrow$ so $\frac{\partial f_1}{\partial x_1}$ is $(+)$;

f_2 goes from $(-)$ to $(+)$

so $f_2 \uparrow$ so $\frac{\partial f_2}{\partial x_1}$ is $(+)$;

<u>For x_2 arrow</u>:

$f_1 = 0$ so $\frac{\partial f_1}{\partial x_1} = 0$;

f_2 goes from $(-)$ to $(+)$

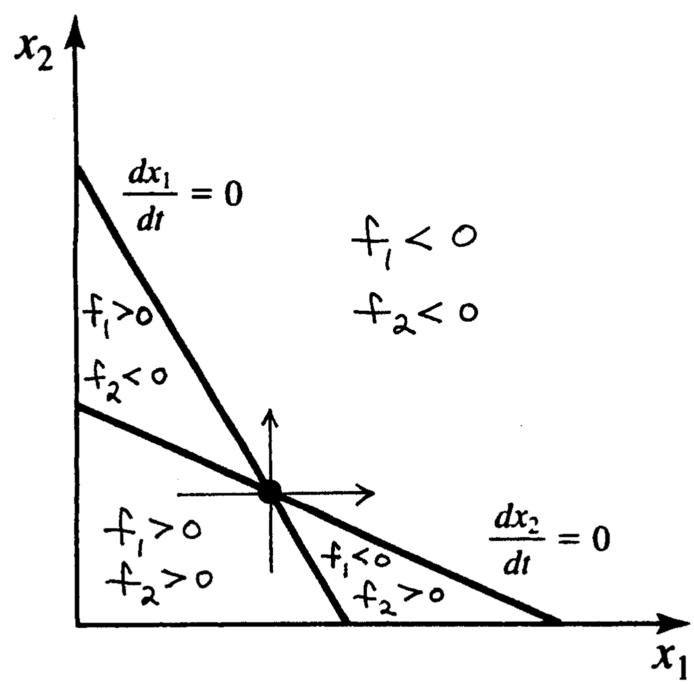
so $f_2 \uparrow$ so $\frac{\partial f_2}{\partial x_2}$ is $(+)$;

the signs for entries of Jacobi Matrix are:

$$Df(a,b) = \begin{bmatrix} + & 0 \\ + & + \end{bmatrix} = A, \text{ then}$$

- tr A is (+), so real parts of both eigenvalues are <u>not</u> negative, so (a,b) is <u>unstable</u> <u>equilibrium</u>.

17.) (again... SEE signchart below.)



For x_1 arrow:

f_1 goes from (+) to (−)
so $f_1 \downarrow$ so $\frac{\partial f_1}{\partial x_1}$ is (−);

f_2 goes from (+) to (−)
so $f_2 \downarrow$ so $\frac{\partial f_2}{\partial x_1}$ is (−);

For x_2 arrow:

f_1 goes from (+) to (−)
so $f_1 \downarrow$ so $\frac{\partial f_1}{\partial x_2}$ is (−);

f_2 goes from (+) to (−)
so $f_2 \downarrow$ so $\frac{\partial f_2}{\partial x_2}$ is (−);

the signs for entries of Jacobi Matrix are:

$$Df(a,b) = \begin{bmatrix} - & - \\ - & - \end{bmatrix} = A, \text{ then}$$

tr A is (−), but det A is (?),
so no conclusion can be made about stability.

23.) $\begin{cases} \dfrac{dx_1}{dt} = 2x_1 - x_1^2 - x_1 x_2 = f_1(x_1, x_2) \\ \dfrac{dx_2}{dt} = x_1 x_2 - x_2 = f_2(x_1, x_2) \end{cases}$

$\rightarrow x_1(2 - x_1 - x_2) = 0 \rightarrow$

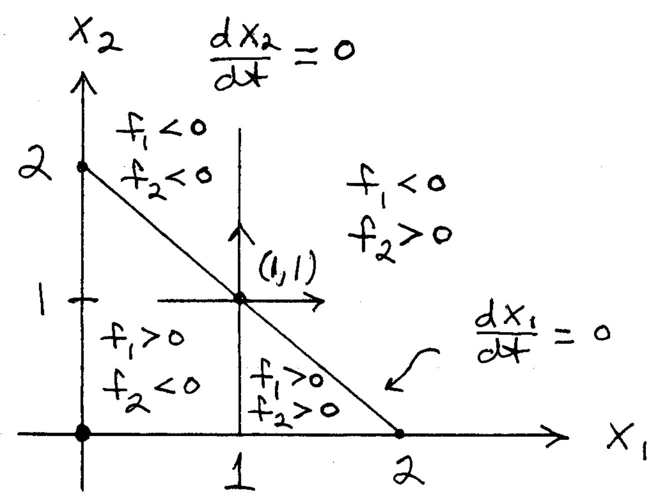
$x_1 = 0$ or $2 - x_1 - x_2 = 0$ (lines);

$\rightarrow x_2(x_1 - 1) = 0 \rightarrow$

$x_2 = 0$ or $x_1 = 1$ (lines);

all equations are zero isoclines

a.)



<u>For x_1 arrow</u>:

f_1 goes from (+) to (−)

so $f_1 \downarrow$ so $\dfrac{\partial f_1}{\partial x_1}$ is (−);

f_2 goes from (−) to (+)

so $f_2 \uparrow$ so $\dfrac{\partial f_2}{\partial x_1}$ is (+);

<u>For X_2 arrow</u> :

f_1 goes from $(+)$ to $(-)$
so $f_1 \downarrow$ so $\frac{\partial f_1}{\partial x_2}$ is $(-)$;

$f_2 = 0$ so $\frac{\partial f_2}{\partial x_2} = 0$; the signs for entries of Jacobi Matrix are

$$Df(1,1) = \begin{bmatrix} - & - \\ + & 0 \end{bmatrix} = A \text{, then}$$

$\mathrm{tr}A$ is $(-)$ and $\det A$ is $(+)$, so real parts of both eigenvalues are negative $\rightarrow$ $(1,1)$ is a stable equilibrium