Section 11.4

15.) $\begin{cases} \frac{dN}{dt} = N - 4PN = N(1-4P) \\ \frac{dP}{dt} = 2PN - 3P = P(2N-3) \end{cases}$

a.) $N(1-4P) = 0 \longrightarrow N=0$ OR $P = 1/4$; and
$P(2N-3) = 0 \rightarrow P=0$ OR $N = 3/2$; then
$N=0$ and $P=0 \rightarrow (0,0)$ is equilibrium
$P = 1/4$ and $N = 3/2 \rightarrow (3/2, 1/4)$ is equilibrium

b.) c.) Let $f(N,P) = \begin{bmatrix} N-4PN \\ 2PN-3P \end{bmatrix} \longrightarrow$ Jacobi Matrix

$$Df(N,P) = \begin{bmatrix} 1-4P & -4N \\ 2P & 2N-3 \end{bmatrix}, \text{ then}$$

$$Df\left(\tfrac{3}{2}, \tfrac{1}{4}\right) = \begin{bmatrix} 0 & -6 \\ 1/2 & 0 \end{bmatrix} = A \longrightarrow$$

$$\det(A - \lambda I) = \det\begin{bmatrix} 0-\lambda & -6 \\ 1/2 & 0-\lambda \end{bmatrix} = \lambda^2 + 3 = 0 \rightarrow$$

$\lambda^2 = -3 \longrightarrow \lambda = \pm\sqrt{3}\,i \longrightarrow \lambda_1 = \sqrt{3}\,i,\ \lambda_2 = -\sqrt{3}\,i$;
since eigenvalues are purely imaginary, this analysis does <u>not</u> lead to a conclusion about the stability of $(3/2, 1/4)$;

$$Df(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} = A \longrightarrow$$

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 0 \\ 0 & -3-\lambda \end{bmatrix}$$

$= (1-\lambda)(-3-\lambda) = 0 \rightarrow \lambda_1 = 1,\ \lambda_2 = -3 \rightarrow$
$(0,0)$ is an <u>unstable</u> equilibrium;

<u>Solve System</u>:

$$\frac{dP}{dN} = \frac{\frac{dP}{dt}}{\frac{dN}{dt}} = \frac{2PN-3P}{N-4PN} = \frac{P(2N-3)}{N(1-4P)} \longrightarrow$$

$$\int \frac{1-4P}{P}\,dP = \int \frac{2N-3}{N}\,dN \longrightarrow$$

$$\int \left(\frac{1}{P} - 4\right) dP = \int \left(2 - \frac{3}{N}\right) dN \longrightarrow$$

$$\boxed{\ln P - 4P = 2N - 3\ln N + C}$$

16.) $\begin{cases} \frac{dN}{dt} = 5N - PN = N(5-P) \\ \frac{dP}{dt} = PN - P = P(N-1) \end{cases}$

a.) $(0,0)$ and $(1,5)$ are equilibria

b.) Let $f(N,P) = \begin{bmatrix} 5N-PN \\ PN-P \end{bmatrix} \rightarrow$

$Df(N,P) = \begin{bmatrix} 5-P & -N \\ P & N-1 \end{bmatrix}$, then

$Df(0,0) = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = A \longrightarrow$

$\det(A-\lambda I) = \det\begin{bmatrix} 5-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}$

$= (5-\lambda)(-1-\lambda) = 0 \longrightarrow \lambda_1 = 5,\ \lambda_2 = -1 \longrightarrow$

$(0,0)$ is <u>unstable</u> equilibrium.

e.) $Df(1,5) = \begin{bmatrix} 0 & -1 \\ 5 & 0 \end{bmatrix} = A \longrightarrow$

$\det(A-\lambda I) = \det\begin{bmatrix} 0-\lambda & -1 \\ 5 & 0-\lambda \end{bmatrix} = \lambda^2 + 5 = 0$

$\longrightarrow \lambda^2 = -5 \longrightarrow \lambda = \pm\sqrt{5}\,i \longrightarrow \lambda_1 = \sqrt{5}\,i,\ \lambda_2 = -\sqrt{5}\,i$;

since eigenvalues are purely imaginary, this analysis does <u>not</u> lead to a conclusion about the stability of $(1,5)$.

<u>Solve System</u>:

$$\frac{dP}{dN} = \frac{\frac{dP}{dt}}{\frac{dN}{dt}} = \frac{P(N-1)}{N(5-P)} \longrightarrow \int \frac{5-P}{P}\,dP = \int \frac{N-1}{N}\,dN$$

$$\longrightarrow \int\left(\frac{5}{P}-1\right)dP = \int\left(1-\frac{1}{N}\right)dN$$

$$\longrightarrow \boxed{5\ln P - P = N - \ln N + C}$$

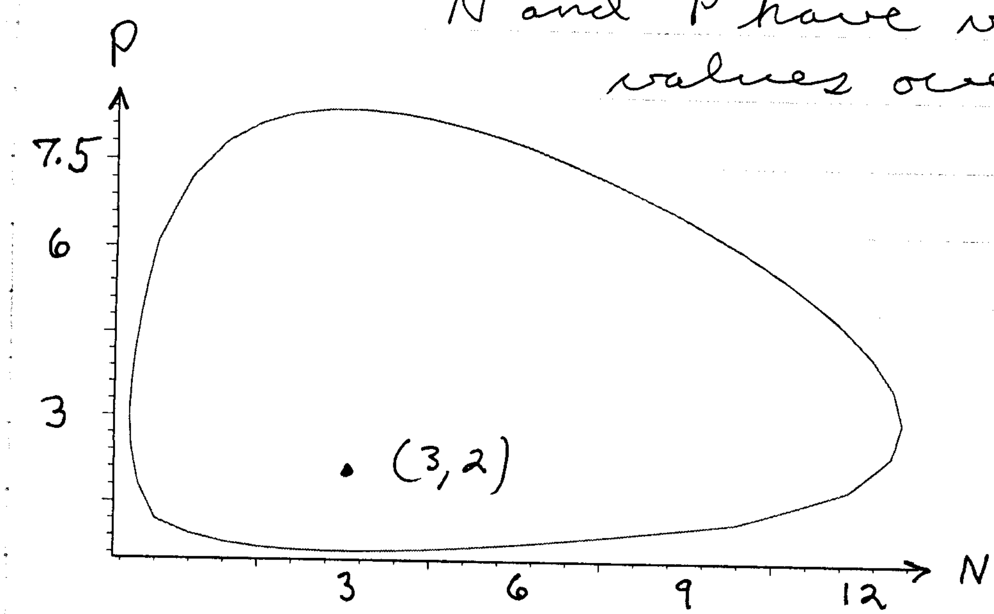
18.) $\begin{cases} \frac{dN}{dt} = 4N - 2PN = N(4-2P) \\ \frac{dP}{dt} = PN - 3P = P(N-3) \end{cases}$

a.) If $N=3$ and $P=2$, then

$\frac{dN}{dt} = 3(4-4) = 0$ and $\frac{dP}{dt} = 2(3-3) = 0$

$\rightarrow$ $N=3$ and $P=2$ for all values of t.

b.) If N and P are dramatically change by outside forces, then we could expect to see the solution curve be one of the closed solution curves enclosing (3,2) and N and P have variable values over time.



19.) $\begin{cases} \frac{dN}{dt} = 3N\left(1-\frac{N}{10}\right) - 2PN \\ \frac{dP}{dt} = PN - 4P \end{cases}$

a.) If no predators, then 1st equation becomes $(P=0)$

$$\frac{dN}{dt} = 3N\left(1-\frac{N}{10}\right) \quad \text{and } N(0) = 9 \ ;$$

$$\frac{dN}{dt} = \frac{3}{10}N(10-N) \rightarrow$$

$$\int \frac{1}{N(10-N)}\, dN = \int \frac{3}{10}\, dt \rightarrow$$

$$\int \left[\frac{A}{N} + \frac{B}{10-N}\right] dN = \frac{3}{10}t + C \rightarrow$$

$($ $A(10-N) + BN = 1$;
Let $N=0$: $10A = 1 \rightarrow A = 1/10$,
Let $N=10$: $10B = 1 \rightarrow B = 1/10$ $) \rightarrow$

$$\int \left[\frac{1/10}{N} + \frac{1/10}{10-N}\right] dN = \frac{3}{10}t + C \rightarrow$$

$$\frac{1}{10}\ln N - \frac{1}{10}\ln(10-N) = \frac{3}{10}t + C \rightarrow$$

$$\ln N - \ln(10-N) = 3t + C \rightarrow$$

$$\ln\left(\frac{N}{10-N}\right) = 3t + C \rightarrow$$

$$e^{\ln\left(\frac{N}{10-N}\right)} = e^{3t+C} = e^{C}e^{3t} = Ce^{3t} \rightarrow$$

$$\frac{N}{10-N} = ce^{3t} \quad \text{and} \quad t=0,\ N=9 \rightarrow$$

$$\frac{9}{1} = c \cdot e^{0} = c(1) = c \longrightarrow c = 9 \rightarrow$$

$$\frac{N}{10-N} = 9e^{3t} \longrightarrow N = 90e^{3t} - 9Ne^{3t} \rightarrow$$

$$N + 9Ne^{3t} = 90e^{3t} \longrightarrow$$

$$N(1+9e^{3t}) = 90e^{3t} \longrightarrow$$

$$\boxed{N = \frac{90e^{3t}}{1+9e^{3t}}}$$

long term behavior:

$$\lim_{t\to\infty} N = \lim_{t\to\infty} \frac{90e^{3t}}{1+9e^{3t}}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{t\to\infty} \frac{270\cancel{e^{3t}}}{27\cancel{e^{3t}}} = 10$$

b.) $\frac{dP}{dt} = PN - 4P = P(N-4) = 0 \rightarrow$

$P=0$ OR $N=4$; and

$\frac{dN}{dt} = 3N\left(1 - \frac{N}{10}\right) - 2PN = 0$;

if $P=0 \rightarrow 3N\left(1-\frac{N}{10}\right)=0 \rightarrow$

$N=0$ or $N=10 \rightarrow$

$(N=0, P=0)$ and $(N=10, P=0)$ are equilibria ; if $N=4 \rightarrow$

$12\left(1-\frac{2}{5}\right)-8P=0 \rightarrow \frac{36}{5}=8P \rightarrow$

$P=\frac{36}{40}=\frac{9}{10} \rightarrow \left(N=4, P=\frac{9}{10}\right)$ is equilibrium ;

$$\frac{\partial f_1}{\partial N}=3N\left(\frac{-1}{10}\right)+3\left(1-\frac{N}{10}\right)-2P$$

$$=-\frac{3}{10}N+3-\frac{3}{10}N-2P=3-\frac{3}{5}N-2P,$$

$$\frac{\partial f_1}{\partial P}=-2N \quad ;$$

$$\frac{\partial f_2}{\partial N}=P \quad , \quad \frac{\partial f_2}{\partial P}=N-4 \quad ;$$

Jacobi matrix is

$$Df(N,P)=\begin{bmatrix} 3-\frac{3}{5}N-2P & -2N \\ P & N-4 \end{bmatrix}=A \quad ;$$

For $(0,0)$: $A=\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} \rightarrow$

$$\det(A-\lambda I)=\det\begin{bmatrix} 3-\lambda & 0 \\ 0 & -4-\lambda \end{bmatrix}$$

$= (3-\lambda)(-4-\lambda) = 0 \rightarrow \lambda_1 = 3,\ \lambda_2 = -4$

so $(0,0)$ is unstable equilibrium (saddle).

<u>For $(10,0)$</u>: $A = \begin{bmatrix} -3 & -20 \\ 0 & 6 \end{bmatrix} \rightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} -3-\lambda & -20 \\ 0 & 6-\lambda \end{bmatrix}$

$= (-3-\lambda)(6-\lambda) = 0 \rightarrow \lambda_1 = 6,\ \lambda_2 = -3$ so $(10,0)$ is unstable equilibrium (saddle).

<u>For $(4, \frac{9}{10})$</u>: $A = \begin{bmatrix} -6/5 & -8 \\ 9/10 & 0 \end{bmatrix} \rightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} -6/5 - \lambda & -8 \\ 9/10 & -\lambda \end{bmatrix} = 0 \rightarrow$

$\lambda^2 + \frac{6}{5}\lambda + \frac{36}{5} = 0 \rightarrow 5\lambda^2 + 6\lambda + 36 = 0 \rightarrow$

$$\lambda = \frac{-6 \pm \sqrt{36 - 4(5)(36)}}{10} = \frac{-6 \pm 6\sqrt{-19}}{10} \rightarrow$$

$\lambda = -\frac{3}{5} \pm \frac{3}{5}\sqrt{19}\, i$ so $(4, \frac{9}{10})$ is stable equilibrium (stable spiral)