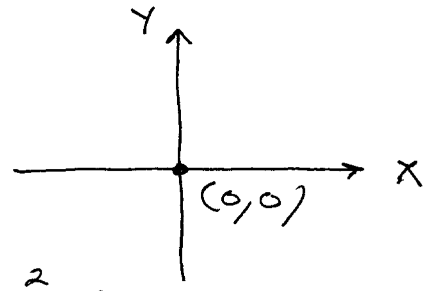
Section 10.2

2.) $\displaystyle\lim_{(x,y)\to(-1,1)} (2xy+3x^2) = 2(-1)(1)+3(-1)^2$
$= -2+3 = 1$

11.) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{2xy-3}{x^2+y^2+1} = \frac{2(0)(1)-3}{0^2+1^2+1} = \frac{-3}{2}$

15.) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-2y^2}{x^2+y^2}$:



along x-axis: $(y=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-2y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-0}{x^2+0}$$

$$= \lim_{x\to 0} 1 = 1 ;$$

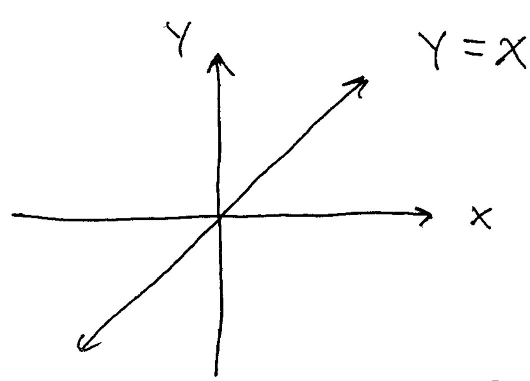
along y-axis: $(x=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2-2y^2}{x^2+y^2} = \lim_{y\to 0} \frac{0-2y^2}{0+y^2}$$

$$= \lim_{y\to 0} (-2) = -2 ;$$ so $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-2y^2}{x^2+y^2}$

$\boxed{\text{DNE}}$.

17.) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+y^2}$:



along x-axis: $(y=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{4xy}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2+0} = \lim_{x\to 0} 0 = 0 ;$$

along Y-axis : $(X=0)$

$$\lim_{\substack{(X,Y)\to(0,0)\\ X=0}} \frac{4XY}{X^2+Y^2} = \lim_{Y\to 0} \frac{0}{0+Y^2}$$

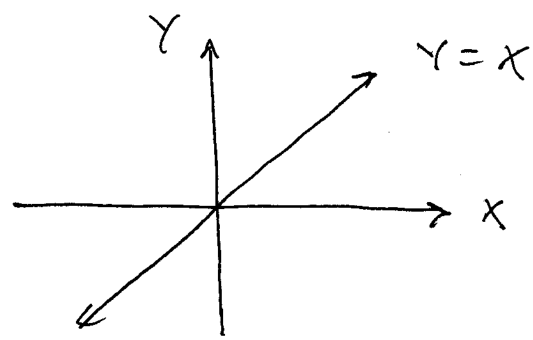
$$= \lim_{Y\to 0} 0 = 0 \quad ;$$

along line $Y=X$:

$$\lim_{\substack{(X,Y)\to(0,0)\\ Y=X}} \frac{4XY}{X^2+Y^2} = \lim_{X\to 0} \frac{4X^2}{X^2+X^2}$$

$$= \lim_{X\to 0} \frac{4X^2}{2X^2} = \lim_{X\to 0} 2 = 2 \quad ; \text{ so}$$

$$\lim_{(X,Y)\to(0,0)} \frac{4XY}{X^2+Y^2} \quad \boxed{\text{DNE}} \; .$$



18.) $\displaystyle\lim_{(X,Y)\to(0,0)} \frac{3XY}{X^2+Y^3}$:

along X-axis : $(Y=0)$

$$\lim_{\substack{(X,Y)\to(0,0)\\ Y=0}} \frac{3XY}{X^2+Y^3}$$

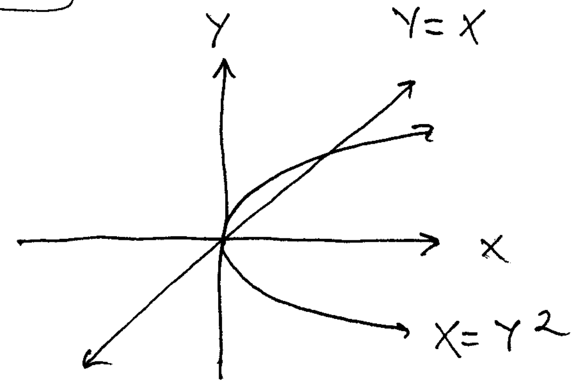
$$= \lim_{X\to 0} \frac{0}{X^2+0} = \lim_{X\to 0} 0 = 0 \quad ;$$

along line $Y=X$:

$$\lim_{\substack{(X,Y)\to(0,0)\\ Y=X}} \frac{3XY}{X^2+Y^3} = \lim_{X\to 0} \frac{3X^2}{X^2+X^3}$$

$$= \lim_{X\to 0} \frac{\cancel{X^2}(3)}{\cancel{X^2}(1+X)} = \frac{3}{1+0} = 3 \quad ; \text{ so}$$

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^3} \quad \boxed{DNE} .$$



20.) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y^2}{x^3+y^6}$:

<u>along line $y=x$</u> :

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{3x^2y^2}{x^3+y^6}$$

$$= \lim_{x\to 0} \frac{3x^2x^2}{x^3+x^6} = \lim_{x\to 0} \frac{\cancel{x^3}(3x)}{\cancel{x^3}(1+x^3)}$$

$$= \frac{0}{1+0} = 0 \quad ;$$

<u>along parabola $x=y^2$</u> :

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} \frac{3x^2y^2}{x^3+y^6} = \lim_{y\to 0} \frac{3(y^2)^2y^2}{(y^2)^3+y^6}$$

$$= \lim_{y\to 0} \frac{3y^6}{y^6+y^6} = \lim_{y\to 0} \frac{3\cancel{y^6}}{2\cancel{y^6}} = \frac{3}{2} \; ; \text{ so}$$

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y^2}{x^3+y^6} \quad \boxed{DNE} .$$

21.) Prove that $f(x,y)=x^2+y^2$ is continuous at $(0,0)$, i.e., prove that $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2) = f(0,0) = 0$:

Let $\varepsilon>0$ be given. Find $\delta>0$ so that if $0 < \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$, then $|(x^2+y^2)-0| = |x^2+y^2| < \varepsilon$.

Then

$$|x^2+y^2| = x^2+y^2$$

$$= \left(\sqrt{x^2+y^2}\right)^2 < \varepsilon \rightarrow$$

iff $\sqrt{x^2+y^2} < \sqrt{\varepsilon}$.

Now choose $\delta = \sqrt{\varepsilon}$ and the result follows. QED

22.) Prove that $f(x,y) = \sqrt{9+x^2+y^2}$ is continuous at $(0,0)$, i.e., prove that $\lim_{(x,y)\to(0,0)} \sqrt{9+x^2+y^2} = f(0,0) = \sqrt{9} = 3$:

Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$, then $\left|\sqrt{9+x^2+y^2} - 3\right| < \varepsilon$.

Then $\left|\sqrt{9+x^2+y^2} - 3\right| = \left|\left(\sqrt{9+x^2+y^2} - 3\right)\frac{\left(\sqrt{9+x^2+y^2}+3\right)}{\left(\sqrt{9+x^2+y^2}+3\right)}\right|$

$$= \frac{|9+x^2+y^2-9|}{\sqrt{9+x^2+y^2}+3}$$

$$= \frac{|x^2+y^2|}{\sqrt{9+x^2+y^2}+3}$$

$$\leq \frac{x^2+y^2}{0+3}$$

$$= \frac{1}{3}\left(\sqrt{x^2+y^2}\right)^2 < \varepsilon$$

iff $\left(\sqrt{x^2+y^2}\right)^2 < 3\varepsilon$

iff $\sqrt{x^2+y^2} < \sqrt{3\varepsilon}$. Now choose $\delta = \sqrt{3\varepsilon}$ and the result follows. QED

25.) $f(x,y) = \begin{cases} \dfrac{2xy}{x^3+xy} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

is not continous at $(0,0)$:

<u>along path $y=0$</u> :

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^3+xy} = \lim_{(x,y)\to(0,0)} \frac{2x(0)}{x^3+x(0)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{0}{x^3} = \lim_{(x,y)\to(0,0)} 0 = 0 \ ;$$

<u>along path $y=x^2$</u> :

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^3+xy} = \lim_{(x,y)\to(0,0)} \frac{2x(x^2)}{x^3+x(x^2)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x^3}{x^3+x^3} = \lim_{(x,y)\to(0,0)} \frac{2x^3}{2x^3}$$

$$= \lim_{(x,y)\to(0,0)} 1 = 1$$; thus $\lim_{(x,y)\to(0,0)} f(x,y)$

DNE so f is <u>not</u> continuous at (0,0).

26.) By problem 20.) $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE so f is <u>not</u> continuous at (0,0).

Math 17C
Kouba
Worksheet 1

1.) Write a precise ϵ/δ proof for each limit.

a.) $\displaystyle\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$

b.) $\displaystyle\lim_{(x,y)\to(0,0)} (2x^2+3y^2) = 0$

c.) $\displaystyle\lim_{(x,y)\to(1,-1)} (x+y) = 0$

d.) $\displaystyle\lim_{(x,y)\to(0,2)} (2x-y+1) = -1$

e.) $\displaystyle\lim_{(x,y)\to(3,2)} xy = 6$

2.) Write a precise ϵ/δ proof for each statement.

a.) $f(x,y) = x^2 - y^2$ is continuous at $(0,0)$.

b.) $f(x,y) = \sqrt{x^2+y^2}$ is continuous at $(2,0)$.

3.) Show that the function $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$

is NOT continuous at $(0,0)$.

4.) Show that the function $f(x,y) = \begin{cases} 1+x+y & , \quad (x,y) \neq (0,0) \\ 2 & , \quad (x,y) = (0,0) \end{cases}$

a.) is NOT continuous at $(0,0)$.

b.) is continuous at $(4,-3)$.

Worksheet 1

1.) a.) $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$,

then $\left|\sqrt{x^2+y^2} - 0\right| = \sqrt{x^2+y^2} < \varepsilon$.

Choose $\delta = \varepsilon$ and the result follows.

QED

b.) $\lim_{(x,y)\to(0,0)} (2x^2+3y^2) = 0$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$,

then $\left|(2x^2+3y^2) - 0\right| = |2x^2+3y^2| < \varepsilon$. Then

$$\begin{aligned} |2x^2+3y^2| &= 2x^2+3y^2 \\ &\leq 3x^2+3y^2 \\ &= 3(x^2+y^2) \\ &= 3\left(\sqrt{x^2+y^2}\right)^2 < \varepsilon \end{aligned}$$

iff $\left(\sqrt{x^2+y^2}\right)^2 < \frac{1}{3}\varepsilon$

iff $\sqrt{x^2+y^2} < \sqrt{\frac{1}{3}\varepsilon}$. Now

choose $\delta = \sqrt{\frac{1}{3}\varepsilon}$ and the result follows.

QED

c.) $\lim_{(x,y)\to(1,-1)} (x+y) = 0$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-1)^2 + (y-(-1))^2} = \sqrt{(x-1)^2+(y+1)^2} < \delta$,

then $|(x+y) - 0| = |x+y| < \varepsilon$. Then

$$|x+y| = |(x-1) + (y+1)|$$

$$\overset{\Delta\text{-inequality}}{\leq} |x-1| + |y+1|$$

$$= \sqrt{(x-1)^2} + \sqrt{(y+1)^2}$$

$$\leq \sqrt{(x-1)^2+(y+1)^2} + \sqrt{(x-1)^2+(y+1)^2}$$

$$= 2\sqrt{(x-1)^2+(y+1)^2} < \varepsilon$$

iff $\sqrt{(x-1)^2+(y+1)^2} < \frac{1}{2}\varepsilon$. Now choose $\delta = \varepsilon/2$ and the result follows. QED

d.) $\lim_{(x,y)\to(0,2)} (2x-y+1) = -1$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{x^2+(y-2)^2} < \delta$, then

$$|(2x-y+1) - (-1)| = |2x-y+2|$$

$$= |2(x) - (y-2) + 2 - 2|$$

$$= |2x - (y-2)|$$

$$\overset{\Delta\text{-inequality}}{\leq} |2x| + |y-2|$$

$= 2|x| + |y-2|$

$= 2\sqrt{x^2} + \sqrt{(y-2)^2}$

$\le 2\sqrt{x^2+(y-2)^2} + \sqrt{x^2+(y-2)^2}$

$= 3\sqrt{x^2+(y-2)^2} < \varepsilon$

iff $\sqrt{x^2+(y-2)^2} < \frac{1}{3}\varepsilon$. Now choose $\delta = \frac{1}{3}\varepsilon$ and the result follows. QED

e.) $\lim_{(x,y)\to(3,2)} xy = 6$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < \sqrt{(x-3)^2+(y-2)^2} < \delta$, then $|xy-6| < \varepsilon$.

Then $|xy-6| = |(x-3)(y-2) - 6 + 2x + 3y - 6|$

$= |(x-3)(y-2) + 2x + 3y - 12|$

$= |(x-3)(y-2) + 2(x-3) + 3(y-2) - 12 + 6 + 6|$

$= |(x-3)(y-2) + 2(x-3) + 3(y-2)|$

Δ-inequality

$\le |(x-3)(y-2)| + |2(x-3)| + |3(y-2)|$

$= |x-3|\,|y-2| + 2|x-3| + 3|y-2|$

$= \sqrt{(x-3)^2}\cdot\sqrt{(y-2)^2} + 2\sqrt{(x-3)^2} + 3\sqrt{(y-2)^2}$

$\le \sqrt{(x-3)^2+(y-2)^2}\cdot\sqrt{(x-3)^2+(y-2)^2} + 2\sqrt{(x-3)^2+(y-2)^2} + 3\sqrt{(x-3)^2+(y-2)^2}$

$$= \left(\sqrt{(x-3)^2+(y-2)^2}\right)^2 + 5\sqrt{(x-3)^2+(y-2)^2}$$

assume $\delta \le 1$

$$\le \sqrt{(x-3)^2+(y-2)^2} + 5\sqrt{(x-3)^2+(y-2)^2}$$

$$= 6\sqrt{(x-3)^2+(y-2)^2} < \varepsilon$$

iff $\sqrt{(x-3)^2+(y-2)^2} < \frac{1}{6}\varepsilon$. Now choose $\delta = \min\{\frac{1}{6}\varepsilon, 1\}$ and the result follows. QED

2.) a.) i.) $f(0,0) = 0^2 - 0^2 = 0$ is defined.

ii.) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$, then $|(x^2-y^2)-0| = |x^2-y^2| < \varepsilon$. Then

$$\begin{aligned} |x^2-y^2| &\le |x^2| + |y^2| \\ &= x^2+y^2 \\ &= \left(\sqrt{x^2+y^2}\right)^2 < \varepsilon \end{aligned}$$

iff $\sqrt{x^2+y^2} < \sqrt{\varepsilon}$. Choose $\delta = \sqrt{\varepsilon}$ and the result follows. QED

iii.) $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

Thus, f is continuous at $(0,0)$.

b.) i.) $f(2,0) = \sqrt{2^2+0^2} = 2$ is defined.

ii.) $\lim_{(x,y)\to(2,0)} f(x,y) = 2$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x-2)^2 + y^2} < \delta$, then

$\left|\sqrt{x^2+y^2} - 2\right| < \varepsilon$. Then

$$\left|\sqrt{x^2+y^2} - 2\right| = \left|(\sqrt{x^2+y^2} - 2)\frac{(\sqrt{x^2+y^2}+2)}{(\sqrt{x^2+y^2}+2)}\right|$$

$$= \left|\frac{x^2+y^2-4}{\sqrt{x^2+y^2}+2}\right|$$

$$\le \frac{|x^2+y^2-4|}{0+2}$$

$$= \frac{1}{2}\left|(x-2)^2 + 4x - 4 + y^2 - 4\right|$$

$$= \frac{1}{2}\left|(x-2)^2 + 4(x-2) + \cancel{8} + y^2 - \cancel{8}\right|$$

$$= \frac{1}{2}\left|(x-2)^2 + 4(x-2) + y^2\right|$$

Δ-inequality

$$\le \frac{1}{2}\left\{|(x-2)^2| + 4|x-2| + |y^2|\right\}$$

$$= \frac{1}{2}\left\{(x-2)^2 + y^2 + 4\sqrt{(x-2)^2}\right\}$$

$$\le \frac{1}{2}\left\{\left(\sqrt{(x-2)^2+y^2}\right)^2 + 4\sqrt{(x-2)^2+y^2}\right\}$$

$$\le \frac{1}{2}\left\{\sqrt{(x-2)^2+y^2} + 4\sqrt{(x-2)^2+y^2}\right\}$$

$$= \frac{5}{2}\sqrt{(x-2)^2+y^2} < \varepsilon$$

assume $\delta \le 1$

iff $\sqrt{(x-2)^2+y^2} < \frac{2}{5}\varepsilon$. Now choose $\delta = \min\{\frac{2}{5}\varepsilon, 1\}$ and the result follows.

QED

3.) i.) $f(0,0)=0$ is defined.

ii.) <u>along $y=0$</u>: $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x(0)}{x^2+(0)^2}$

$= \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = \lim_{(x,y)\to(0,0)} 0 = 0$;

<u>along $y=x$</u>: $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x(x)}{x^2+(x)^2}$

$= \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \lim_{(x,y)\to(0,0)} \frac{1}{2} = \frac{1}{2}$;

so $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE and f is NOT continuous at $(0,0)$.

4.) a.) i.) $f(0,0)=2$ is defined.

ii.) $\lim_{(x,y)\to(0,0)} (1+x+y) = 1+0+0 = 1$.

iii.) $\lim_{(x,y)\to(0,0)} f(x,y) = 1 \neq 2 = f(0,0)$,

so f is NOT continuous at $(0,0)$.

b.) i.) $f(4,-3) = 1+(4)+(-3) = 2$ is defined.

ii.) $\lim_{(x,y)\to(4,-3)} (1+x+y) = 1+(4)+(-3) = 2$.

iii.) $\lim_{(x,y)\to(4,-3)} f(x,y) = 2 = f(4,-3)$.

So f is continuous at $(4,-3)$.