Section 10.5

2.) $f(x,y) = e^x \sin y, \quad x = t, \quad y = t^3 \xrightarrow{D}$

$$\frac{dw}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$= e^x \sin y \cdot (1) + e^x \cos y \cdot (3t^2)$$

and $t = 1 \rightarrow x = 1, \ y = 1$ so

$$\frac{dw}{dt} = e^1 \sin 1 + e^1 \cos 1 \cdot (3) = e \sin 1 + 3e \cos 1$$

3.) $f(x,y) = \sqrt{x^2+y^2}, \quad x = t, \quad y = \sin t \xrightarrow{D}$

$$\frac{dw}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$= \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}x \cdot (1) + \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}y \cdot \cos t$$

and $t = \frac{\pi}{3} \rightarrow x = \frac{\pi}{3}, \ y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ so

$$\frac{dw}{dt} = \frac{\pi/3}{\sqrt{\frac{\pi^2}{9} + \frac{3}{4}}} + \frac{\frac{\sqrt{3}}{2} \overset{1/2}{\cancel{\cos}} \pi/3}{\sqrt{\frac{\pi^2}{9} + \frac{3}{4}}} = \frac{\frac{\pi}{3} + \frac{\sqrt{3}}{4}}{\sqrt{\frac{\pi^2}{9} + \frac{3}{4}}}$$

4.) $f(x,y) = \ln(xy - x^2), \quad x = t^2, \quad y = t \xrightarrow{D}$

$$\frac{dw}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$= \frac{y - 2x}{xy - x^2} \cdot (2t) + \frac{x}{xy - x^2} \cdot (1) \quad \text{and}$$

$t = 5 \rightarrow x = 25, \ y = 5$ so

$$\frac{dw}{dt} = \frac{5 - 50}{125 - 625}(10) + \frac{25}{125 - 625}$$

$$= \frac{-450}{-500} + \frac{25}{-500} = \frac{425}{500} = 0.85$$

7.) $z = f(x,y)$, $x = u(t)$, $y = v(t)$ $\xrightarrow{D}$

$$\frac{dz}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} = f_x \cdot u'(t) + f_y \cdot v'(t)$$

8.) $w = e^{f(x,y)}$, $x = u(t)$, $y = v(t)$ $\xrightarrow{D}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= e^{f(x,y)} \cdot f_x \cdot u'(t) + e^{f(x,y)} \cdot f_y \cdot v'(t)$$

11.) $\ln(x^2+y^2) = 3xy \rightarrow$

$F(x,y) = \ln(x^2+y^2) - 3xy = 0$ $\xrightarrow{D}$

$F_x = \dfrac{2x}{x^2+y^2} - 3y$, $F_y = \dfrac{2y}{x^2+y^2} - 3x$ so

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{3y - \frac{2x}{x^2+y^2}}{\frac{2y}{x^2+y^2} - 3x} \cdot \frac{x^2+y^2}{x^2+y^2}$$

$$= \frac{3y(x^2+y^2) - 2x}{2y - 3x(x^2+y^2)} = \frac{3x^2y + 3y^3 - 2x}{2y - 3x^3 - 3xy^2}$$

14.) $y = \arctan x \rightarrow F(x,y) = y - \arctan x = 0$

$\xrightarrow{D}$ $F_x = \dfrac{-1}{1+x^2}$, $F_y = 1$ so

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{\frac{1}{1+x^2}}{1} = \frac{1}{1+x^2}$$

17.) $f(x,y) = x^3y^2$ $\xrightarrow{D}$ $f_x = 3x^2y^2$,

$f_y = 2x^3y$ so $\nabla f = \begin{bmatrix} 3x^2y^2 \\ 2x^3y \end{bmatrix}$

18.) $f(x,y) = \dfrac{xy}{x^2+y^2} \xrightarrow{D}$

$$f_x = \frac{(x^2+y^2)\cdot y - xy\cdot 2x}{(x^2+y^2)^2} = \frac{x^2y + y^3 - 2x^2y}{(x^2+y^2)^2}$$

$$= \frac{y^3 - x^2y}{(x^2+y^2)^2} \quad ;$$

$$f_y = \frac{(x^2+y^2)\cdot x - xy\cdot(2y)}{(x^2+y^2)^2} = \frac{x^3 + xy^2 - 2xy^2}{(x^2+y^2)^2}$$

$$= \frac{x^3 - xy^2}{(x^2+y^2)^2} \quad \text{so}$$

$$\nabla f = \begin{bmatrix} \dfrac{y^3 - x^2y}{(x^2+y^2)^2} \\ \dfrac{x^3 - xy^2}{(x^2+y^2)^2} \end{bmatrix}$$

22.) $f(x,y) = \tan\left(\dfrac{x-y}{x+y}\right) \xrightarrow{D}$

$$f_x = \sec^2\left(\frac{x-y}{x+y}\right)\cdot \frac{(x+y)(1) - (x-y)(1)}{(x+y)^2}$$

$$= \sec^2\left(\frac{x-y}{x+y}\right)\cdot \frac{2y}{(x+y)^2} \quad ;$$

$$f_y = \sec^2\left(\frac{x-y}{x+y}\right)\cdot \frac{(x+y)(-1) - (x-y)(1)}{(x+y)^2}$$

$$= \sec^2\left(\frac{x-y}{x+y}\right)\cdot \frac{-2x}{(x+y)^2} \quad \text{so}$$

$$\nabla f = \begin{bmatrix} \sec^2\left(\frac{x-y}{x+y}\right) \cdot \frac{2y}{(x+y)^2} \\ \sec^2\left(\frac{x-y}{x+y}\right) \cdot \frac{-2x}{(x+y)^2} \end{bmatrix}$$

25.) $f(x,y) = \sqrt{2x^2+y^2}$, pt. $(1,2)$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $u = \frac{1}{|x|}x = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$;

$$f_x = \frac{1}{\cancel{2}}(2x^2+y^2)^{-1/2} \cdot \overset{2}{\cancel{4}}x = \frac{2x}{\sqrt{2x^2+y^2}},$$

$$f_y = \frac{1}{\cancel{2}}(2x^2+y^2)^{-1/2} \cdot \cancel{2}y = \frac{y}{\sqrt{2x^2+y^2}}; \text{ then}$$

$$D_u f(1,2) = \nabla f(1,2) \cdot u = \begin{bmatrix} f_x(1,2) \\ f_y(1,2) \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{2}{\sqrt{12}} + \frac{2}{\sqrt{12}} = \frac{2}{2\sqrt{3}} + \frac{2}{2\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}$$

27.) $f(x,y) = e^{x+y}$, pt. $(0,0)$, $x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

so $u = \frac{1}{|x|}x = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$;

$f_x = e^{x+y}$, $f_y = e^{x+y}$ so $\nabla f = \begin{bmatrix} e^{x+y} \\ e^{x+y} \end{bmatrix}$;

then

$$D_u f(0,0) = \nabla f(0,0) \cdot u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

28.) $f(x,y) = x^3y^2$, pt. $(2,3)$, $X = \begin{bmatrix}-2\\1\end{bmatrix}$

so $u = \frac{1}{|X|}X = \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$;

$f_x = 3x^2y^2$, $f_y = 2x^3y$ so $\nabla f = \begin{bmatrix}3x^2y^2\\2x^3y\end{bmatrix}$;

then

$D_u f(2,3) = \nabla f(2,3) \cdot u = \begin{bmatrix}108\\48\end{bmatrix} \cdot \begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$

$= \frac{-216}{\sqrt{5}} + \frac{48}{\sqrt{5}} = \frac{-168}{\sqrt{5}}$.

31.) $f(x,y) = 2x^2y - 3x$, pt. $(2,1)$, vector

$\overrightarrow{PQ} = \begin{bmatrix}3-2\\2-1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$ so $u = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$,

$f_x = 4xy - 3$, $f_y = 2x^2$ so

$\nabla f = \begin{bmatrix}4xy-3\\2x^2\end{bmatrix}$; then

$D_u f(2,1) = \nabla f(2,1) \cdot u = \begin{bmatrix}5\\8\end{bmatrix} \cdot \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$= \frac{5}{\sqrt{2}} + \frac{8}{\sqrt{2}} = \frac{13}{\sqrt{2}}$.

34.) $f(x,y) = e^{x-y}$, pt. $(2,2)$, vector

$\overrightarrow{PQ} = \begin{bmatrix}1-2\\-1-2\end{bmatrix} = \begin{bmatrix}-1\\-3\end{bmatrix}$ so $u = \frac{1}{\sqrt{10}}\begin{bmatrix}-1\\-3\end{bmatrix} = \begin{bmatrix}-1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix}$,

$f_x = e^{x-y}$, $f_y = e^{x-y} \cdot (-1) = -e^{x-y}$, so

$\nabla f = \begin{bmatrix} e^{x-y} \\ -e^{x-y} \end{bmatrix}$, then

$D_u f(2,2) = \nabla f(2,2) \cdot u = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$

$= \dfrac{-1}{\sqrt{10}} + \dfrac{3}{\sqrt{10}} = \dfrac{2}{\sqrt{10}}$.

35.) $f(x,y) = 3xy - x^2 \xrightarrow{D} f_x = 3y - 2x$, $f_y = 3x$ so $\nabla f = \begin{bmatrix} 3y-2x \\ 3x \end{bmatrix}$; so at pt. $(-1,1)$ $f \uparrow$ most rapidly in the direction of the gradient vector

$\nabla f(-1,1) = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$.

37.) $f(x,y) = \sqrt{x^2 - y^2} \xrightarrow{D}$

$f_x = \frac{1}{2}(x^2-y^2)^{-1/2} \cdot 2x = \dfrac{x}{\sqrt{x^2-y^2}}$,

$f_y = \frac{1}{2}(x^2-y^2)^{-1/2} \cdot (-2y) = \dfrac{-y}{\sqrt{x^2-y^2}}$ so

$\nabla f = \begin{bmatrix} \dfrac{x}{\sqrt{x^2-y^2}} \\ \dfrac{-y}{\sqrt{x^2-y^2}} \end{bmatrix}$; at pt. $(5,3)$ $f \uparrow$ most rapidly in the direction of the gradient vector

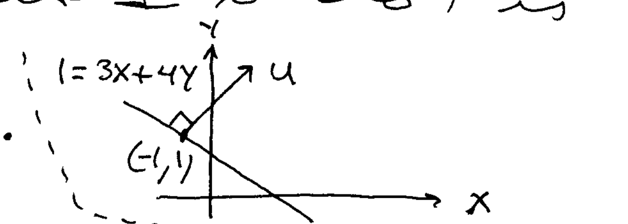
$\nabla f(5,3) = \begin{bmatrix} 5/4 \\ -3/4 \end{bmatrix}$

39.) $f(x,y) = 3x + 4y \xrightarrow{D} f_x = 3,\ f_y = 4$ and

$\nabla f = \begin{bmatrix}3\\4\end{bmatrix}$ so $\nabla f(-1,1) = \begin{bmatrix}3\\4\end{bmatrix}$;

$f(-1,1) = 3(-1) + 4(1) = 1$ so gradient vector $\begin{bmatrix}3\\4\end{bmatrix}$ is $\perp$ to level curve $1 = 3x + 4y$ at pt. $(-1,1)$, so unit $\perp$ vector is

$$u = \frac{1}{\sqrt{25}}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}3/5\\4/5\end{bmatrix}.$$

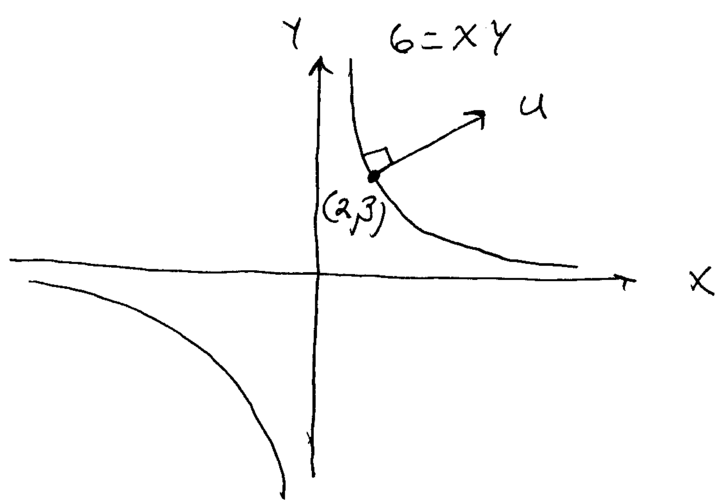


42.) $f(x,y) = xy \xrightarrow{D} f_x = y,\ f_y = x$ and

$\nabla f = \begin{bmatrix}y\\x\end{bmatrix}$ so $\nabla f(2,3) = \begin{bmatrix}3\\2\end{bmatrix}$;

$f(2,3) = 6$ so gradient vector $\begin{bmatrix}3\\2\end{bmatrix}$ is $\perp$ to level curve $6 = xy$ at pt. $(2,3)$, so unit $\perp$ vector is

$$u = \frac{1}{\sqrt{13}}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}3/\sqrt{13}\\2/\sqrt{13}\end{bmatrix}.$$



44.) $f(x,y) = x^2 - y^2 \xrightarrow{D} f_x = 2x,\ f_y = -2y$

so $\nabla f = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$; so at pt. $(2,3)$ $f \downarrow$

most rapidly in the OPPOSITE direction of the gradient vector

$\nabla f(2,3) = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$, i.e., in the

direction of $\quad -\nabla f(2,3) = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$.