Section 16.6

1.) $f(x,y) = x^2 + y^2 - 2x \xrightarrow{D}$

$f_x = 2x - 2 = 0 \rightarrow x = 1$,

$f_y = 2y = 0 \rightarrow y = 0$, so crit. pt. is $\boxed{(1,0)}$:

$f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$; so

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (2)(2) - (0)^2 > 0$

and $f_{xx} = 2 > 0$ (U) so $\boxed{(1,0)}$ determines a min. value of $f(1,0) = -1$.

3.) $f(x,y) = x^2 y - 4x^2 - 4y \xrightarrow{D}$

$f_x = 2xy - 8x = 2x(y-4) = 0 \rightarrow x = 0$ or $y = 4$,

$f_y = x^2 - 4 = (x-2)(x+2) = 0 \rightarrow x = 2$ or $x = -2$;

so critical pts. are $\boxed{(2,4)}$ and $\boxed{(-2,4)}$:

$f_{xx} = 2y - 8$, $f_{yy} = 0$, $f_{xy} = 2x$; so

For $(2,4)$: $D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (0)(0) - (4)^2 = -16 < 0$

so $(2,4)$ determines a saddle point at $z = -16$.

For $(-2,4)$: $D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (0)(0) - (-4)^2 = -16 < 0$

so $(-2,4)$ determines a saddle point at $z = -16$.

4.) $f(x,y) = xy - 2y^2 \xrightarrow{D}$

$f_x = y = 0 \rightarrow y = 0$,

$f_y = x - 4y = 0 \rightarrow x = 4y$; so

$\boxed{(0,0)}$ is critical pt. :

$f_{xx} = 0$, $f_{yy} = -4$, $f_{xy} = 1$; then

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (0)(-4) - (1)^2 = -1 < 0$,

so $(0,0)$ determines a <u>saddle</u> pt. at $z = 0$.

6.) $f(x,y) = x - x^2 + xy \xrightarrow{D}$

$f_x = 1 - 2x + y = 0 \longrightarrow y = 2x - 1$,

$f_y = x = 0$, so critical pt. is

$\boxed{(0,-1)}$:

$f_{xx} = -2$, $f_{yy} = 0$, $f_{xy} = 1$ then

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (-2)(0) - (1)^2 = -1 < 0$

so $(0,-1)$ determines a <u>saddle</u> pt. at $z = 0$.

7.) $f(x,y) = e^{-x^2 - y^2} \xrightarrow{D}$

$f_x = -2x\, e^{-x^2 - y^2} = 0 \longrightarrow x = 0$,

$f_y = -2y\, e^{-x^2 - y^2} = 0 \longrightarrow y = 0$, so

$\boxed{(0,0)}$ is critical pt. :

$f_{xx} = -2x e^{-x^2 - y^2} \cdot (-2x) + (-2) e^{-x^2 - y^2}$,

$f_{yy} = -2y e^{-x^2 - y^2} \cdot (-2y) + (-2) e^{-x^2 - y^2}$,

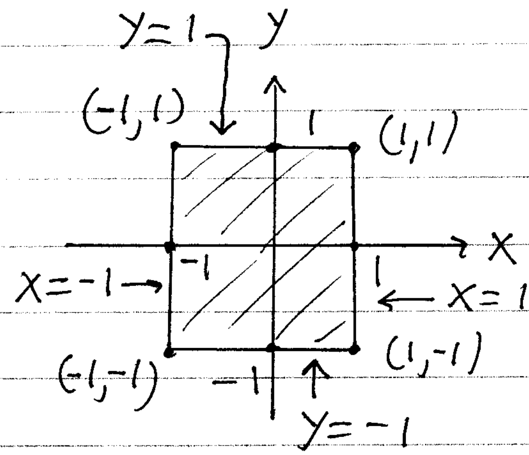
$f_{xy} = -2x\, e^{-x^2 - y^2} \cdot (-2y)$; then

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (-2)(-2) - (0)^2 = 4 > 0$ and

$f_{xx} = -2 < 0$ ($\cap$), so $(0,0)$ determines a maximum value of $f(0,0) = e^0 = 1$.

(PLANE)

14.) $f(x,y) = 3 - x + 2y \rightarrow$ $f_x = -1$, $f_y = 2$ so no critical points;



corners	z-values	
(1,1)	4	
(1,-1)	0	← ABS MIN
(-1,-1)	2	
(-1,1)	6	← ABS MAX

15.) $f(x,y) = x^2 - y^2 \longrightarrow f_x = 2x = 0 \rightarrow x = 0$ and $f_y = -2y = 0 \longrightarrow y = 0$ so $(0,0)$ is critical point; and

along $x = 1$: $f(x,y) = 1 - y^2 \rightarrow f_y = -2y = 0 \rightarrow y = 0$ so $(1,0)$ is crit. pt.;

along $x = -1$: $f(x,y) = 1 - y^2 \rightarrow f_y = -2y = 0 \rightarrow y = 0$ so $(-1,0)$ is crit. pt.;

along $y = 1$: $f(x,y) = x^2 - 1 \rightarrow f_x = 2x = 0 \rightarrow x = 0$ so $(0,1)$ is crit. pt.;

along $y=-1$: $f(x,y) = x^2-1 \rightarrow f_x = 2x = 0$
$\rightarrow x=0$ so $(0,-1)$ is crit. pt. ;

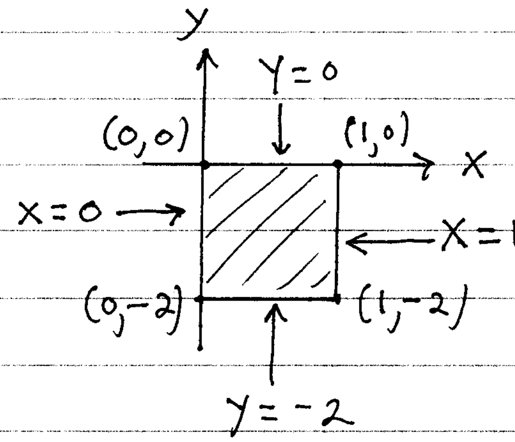
	(x,y)	z-values	
	$(0,0)$	0	
edges	$(1,0)$	1	$\leftarrow$ ABS MAX
	$(-1,0)$	1	$\leftarrow$ ABS MAX
	$(0,1)$	-1	$\leftarrow$ ABS MIN
	$(0,-1)$	-1	$\leftarrow$ ABS MIN
corners	$(1,1)$	0	
	$(1,-1)$	0	
	$(-1,-1)$	0	
	$(-1,1)$	0	

16.) $f(x,y) = 1-x^2+2y^3 \rightarrow$
$f_x = -2x = 0 \rightarrow x=0$, $f_y = 6y^2 = 0 \rightarrow y=0$
so $(0,0)$ is crit. pt. ;
along $x=1$: $f(x,y) = 2y^3 \rightarrow f_y = 6y^2 = 0$
$\rightarrow y=0$ so $(1,0)$ is crit. pt. ;
along $x=-1$: $f(x,y) = 2y^3 \rightarrow f_y = 6y^2 = 0$
$\rightarrow y=0$ so $(-1,0)$ is crit pt. ;
along $y=1$: $f(x,y) = 3-x^2 \rightarrow f_x = -2x = 0 \rightarrow$
$x=0$ so $(0,1)$ is crit. pt. ;
along $y=-1$: $f(x,y) = -1-x^2 \rightarrow f_x = -2x = 0 \rightarrow$
$x=0$ so $(0,-1)$ is crit. pt. ;

	(x,y)	z-values	
	$(0,0)$	1	
edges	$(1,0)$	0	
edges	$(-1,0)$	0	
edges	$(0,1)$	3	$\leftarrow$ ABS MAX
edges	$(0,-1)$	-1	
corners	$(1,1)$	2	
corners	$(1,-1)$	-2	$\leftarrow$ ABS MIN
corners	$(-1,-1)$	-2	$\leftarrow$ ABS MIN
corners	$(-1,1)$	2	

17.) $f(x,y) = x^2 + y^2 - x + 2y$

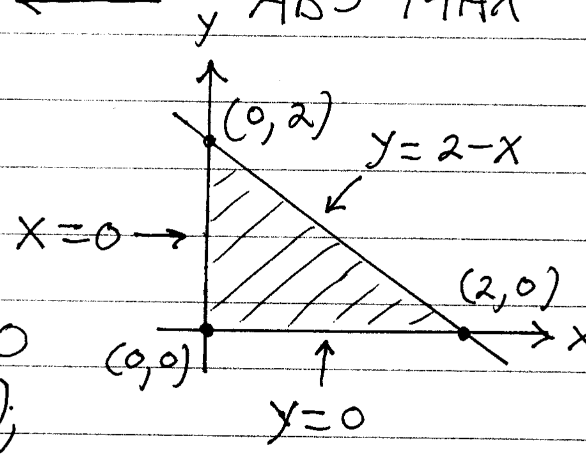
$\rightarrow f_x = 2x - 1 = 0 \rightarrow$
$x = 1/2$ and
$f_y = 2y + 2 = 0 \rightarrow$
$y = -1$ so
$(1/2, -1)$ is crit. pt. ;



along $y=0$: $f(x,y) = x^2 - x \rightarrow f_x = 2x - 1 = 0$
$\rightarrow x = 1/2$ so $(\frac{1}{2}, 0)$ is crit. pt. ;
along $x=1$: $f(x,y) = y^2 + 2y \rightarrow f_y = 2y + 2 = 0$
$\rightarrow y = -1$ so $(1,-1)$ is crit. pt. ;
along $y=-2$: $f(x,y) = x^2 - x \rightarrow f_x = 2x - 1 = 0$
$\rightarrow x = 1/2$ so $(\frac{1}{2}, -2)$ is crit. pt. ;
along $x=0$: $f(x,y) = y^2 + 2y \rightarrow f_y = 2y + 2 = 0$
$\rightarrow y = -1$ so $(0,-1)$ is crit. pt. ;

	(x,y)	z-values	
	$(\frac{1}{2},-1)$	$-5/4$	$\leftarrow$ ABS MIN
edges	$(\frac{1}{2},0)$	$-1/4$	
edges	$(1,-1)$	-1	
edges	$(1/2,-2)$	$-1/4$	
edges	$(0,-1)$	-1	
corners	$(0,0)$	0	$\leftarrow$ ABS MAX
corners	$(1,0)$	0	$\leftarrow$ ABS MAX
corners	$(0,-2)$	0	$\leftarrow$ ABS MAX
corners	$(1,-2)$	0	$\leftarrow$ ABS MAX

19.) $f(x,y) = 2xy - x^2y - xy^2$



$\rightarrow f_x = 2y - 2xy - y^2$

$= y(2-2x-y) = 0$

$\rightarrow y=0$ OR $2-2x-y=0$

so $\boxed{y=0}$ OR $\boxed{y=2-2x}$;

and $f_y = 2x - x^2 - 2xy$

$= x(2-x-2y) = 0 \rightarrow$

$x=0$ OR $2-x-2y=0$ so $\boxed{x=0}$ OR $\boxed{x=2-2y}$;

$\underline{y=0}$ and $\underline{x=0}$ so $(0,0)$ is crit. pt. ;

$\underline{y=0}$ and $\underline{x=2-2y}$ so $x=2$ so $(2,0)$ is crit. pt. ;

$\underline{y=2-2x}$ and $\underline{x=0}$ so $y=2$ so $(0,2)$ is crit. pt. ;

$\underline{y=2-2x}$ and $\underline{x=2-2y} \rightarrow x = 2-2(2-2x) \rightarrow$

$x = 2-4+4x \rightarrow 2 = 3x \rightarrow x = 2/3$ and $y = 2/3$

so $(2/3, 2/3)$ is crit. pt. ;

$\underline{\text{along } y=0}$: $f(x,y) = 0$

$\underline{\text{along } x=0}$: $f(x,y)=0$

$\underline{\text{along } y=2-x}$:

$$f(x,y) = 2x(2-x) - x^2(2-x) - x(2-x)^2$$
$$= 4x - 2x^2 - 2x^2 + x^3 - x(x^2-4x+4)$$
$$= x^3 - 4x^2 + 4x - x^3 + 4x^2 - 4x = 0;$$

	(x,y)	z-values	
	$(0,0)$	0	
	$(2,0)$	0	
	$(0,2)$	0	
	$(2/3, 2/3)$	$8/27$	← ABS MAX
corners	$(0,2)$	0	
corners	$(2,0)$	0	
corners	$(0,0)$	0	

ABS MIN occurs at each point on boundary of triangle.

28.) assume $X+Y+Z=60$, maximize $P=XYZ$; and $Z=60-X-Y$ so

$P=XY(60-X-Y)=60XY-X^2Y-XY^2 \rightarrow$

$\boxed{P=60XY-X^2Y-XY^2}$; $\xrightarrow{D}$

$P_X=60Y-2XY-Y^2=Y(60-2X-Y)=0$

$\rightarrow Y=0$ or $60-2X-Y=0$

$\rightarrow \underline{Y=0}$ or $\underline{Y=60-2X}$;

$P_Y=60X-X^2-2XY=X(60-X-2Y)=0$

$\rightarrow X=0$ or $60-X-2Y=0$

$\rightarrow \underline{X=0}$ or $\underline{X=60-2Y}$; so

$X=0, Y=0$ or $X=0, Y=60-2(0)$ or

$Y=0, X=60-2(0)$ or $Y=60-2X$,

$X=60-2Y \rightarrow X=60-2(60-2X)=60-120+4X$

$\rightarrow 3X=60 \rightarrow X=20, Y=20$; ~~then~~

critical points are

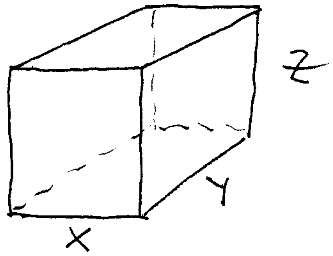
$(0,0)$, $(0,60)$, $(60,0)$, $(20,20)$:

$X=0, Y=0, Z=60 \rightarrow P=0$,

$X=0, Y=60, Z=0 \rightarrow P=0$,

$X=60, Y=0, Z=0 \rightarrow P=0$,

$X=20, Y=20, Z=20 \rightarrow$ $\boxed{P=8000}$ MAX

29.) 

Given surface area

$2XY + 2XZ + 2YZ = 48\ m^2$

$\rightarrow XY + XZ + YZ = 24$

$\rightarrow (X+Y)Z = 24 - XY \rightarrow \boxed{Z = \dfrac{24 - XY}{X+Y}}$;

maximize volume

$V = XYZ = XY \cdot \dfrac{24 - XY}{X+Y} = \dfrac{24XY - X^2Y^2}{X+Y} \rightarrow$

$\boxed{V = \dfrac{24XY - X^2Y^2}{X+Y}}$;

$V_X = \dfrac{(X+Y)(24Y - 2XY^2) - (24XY - X^2Y^2)(1)}{(X+Y)^2}$

$= \dfrac{\cancel{24XY} + 24Y^2 - 2X^2Y^2 - 2XY^3 - \cancel{24XY} + X^2Y^2}{(X+Y)^2}$

$= \dfrac{24Y^2 - X^2Y^2 - 2XY^3}{(X+Y)^2}$

$= \dfrac{Y^2(24 - X^2 - 2XY)}{(X+Y)^2} = 0 \rightarrow$

$Y^2 = 0$ (NO) or $24 - X^2 - 2XY = 0$

$\longrightarrow \boxed{24 - X^2 = 2XY}$; and

$$V_Y = \frac{(X+Y)(24X - 2X^2Y) - (24XY - X^2Y^2)(1)}{(X+Y)^2}$$

$$= \frac{24X^2 + \cancel{24XY} - 2X^3Y - 2X^2Y^2 - \cancel{24XY} + X^2Y^2}{(X+Y)^2}$$

$$= \frac{24X^2 - 2X^3Y - X^2Y^2}{(X+Y)^2}$$

$$= \frac{X^2(24 - 2XY - Y^2)}{(X+Y)^2} = 0 \longrightarrow$$

$X^2 = 0$ (NO) or $24 - 2XY - Y^2 = 0 \rightarrow$

$\boxed{24 - Y^2 = 2XY}$; combining

the boxed equations gives

$$24 - X^2 = 2XY = 24 - Y^2 \longrightarrow$$
$$24 - X^2 = 24 - Y^2 \longrightarrow$$
$$X^2 = Y^2 \longrightarrow \boxed{X = Y}$$; then (SUB)

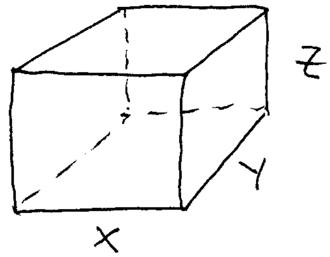
$$24 - Y^2 = 2(Y)Y \longrightarrow 24 = 3Y^2 \longrightarrow$$
$$Y^2 = 8 \longrightarrow$$

$$Y = \sqrt{8},\ X = \sqrt{8},\ Z = \sqrt{8}$$

and max. volume

$$V = 8\sqrt{8}$$

31.)



Given volume

$XYZ = 216 \text{ m.}^3 \rightarrow$

$\boxed{Z = \frac{216}{XY}}$;

minimize surface area

$S = 2XY + 2XZ + 2YZ$

$= 2XY + (2X + 2Y)Z$

$= 2XY + (2X+2Y) \cdot \frac{216}{XY}$

$= 2XY + \frac{432}{Y} + \frac{432}{X} \longrightarrow$

$\boxed{S = 2XY + \frac{432}{Y} + \frac{432}{X}}$; ~~then~~

$S_X = 2Y - \frac{432}{X^2} = 0 \rightarrow \boxed{Y = \frac{216}{X^2}}$;

$S_Y = 2X - \frac{432}{Y^2} = 0 \rightarrow \boxed{X = \frac{216}{Y^2}}$; so

$X = \frac{216}{Y^2} = \frac{216}{\left(\frac{216}{X^2}\right)^2} = \cancel{216} \cdot \frac{X^4}{216^{\cancel{2}}} \longrightarrow$

$216X = X^4 \longrightarrow X^4 - 216X = 0 \rightarrow$

$X(X^3 - 216) = 0 \rightarrow X = 0$ (NO) or

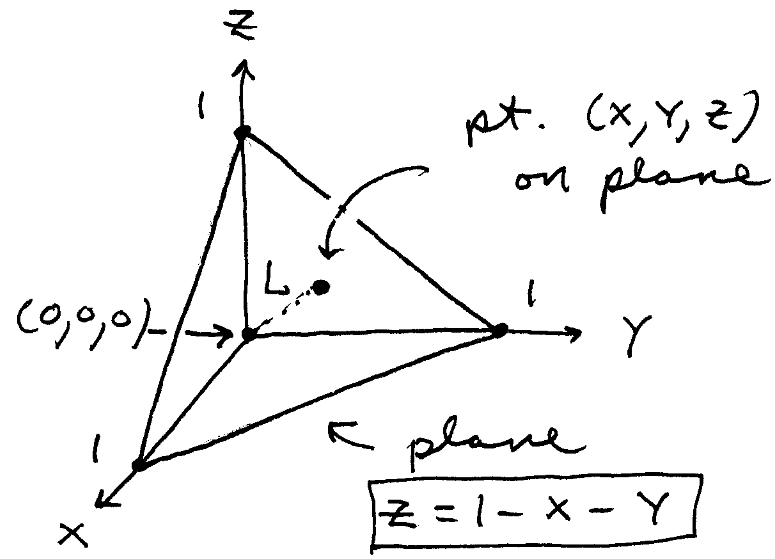
$X = 6$ m., $Y = 6$ m., $Z = 6$ m.

and min. surface area

$S = 216$ m.^{2}

33.) minimize distance

$L = \sqrt{X^2 + Y^2 + Z^2}$



$\boxed{Z = 1 - X - Y}$

(SUB)

$\rightarrow L = \sqrt{X^2 + Y^2 + (1 - X - Y)^2}$; now find critical point :

$\xrightarrow{D} L_X = \frac{1}{2}(\text{—})^{-1/2} \cdot \{2X + 2(1 - X - Y)\cdot(-1)\} = 0$

and $L_Y = \frac{1}{2}(\text{—})^{-1/2} \cdot \{2Y + 2(1 - X - Y)\cdot(-1)\} = 0$

$\rightarrow \begin{cases} 2X - 2 + 2X + 2Y = 0 \\ 2Y - 2 + 2X + 2Y = 0 \end{cases} \rightarrow \begin{cases} 4X + 2Y = 2 \\ 4Y + 2X = 2 \end{cases}$

$\rightarrow \begin{cases} 2X + Y = 1 \\ 2Y + X = 1 \end{cases} \rightarrow \boxed{Y = 1 - 2X} \rightarrow 2(1 - 2X) + X = 1$

$\rightarrow 2 - 4X + X = 1 \rightarrow 3X = 1 \rightarrow$

$X = 1/3$, $Y = \frac{1}{3}$, $Z = 1/3$