Math 17C

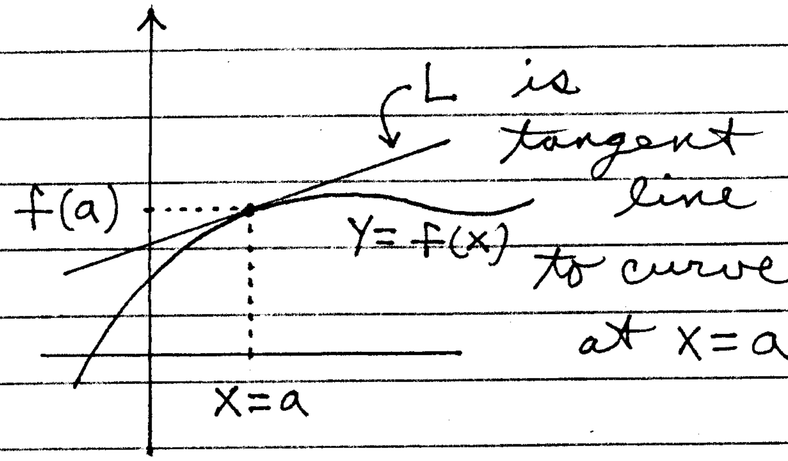
Kouba

Linearization and the Jacobi Matrix

Recall: 1.) (from Math 17A) For function $y=f(x)$ with $f: \mathbb{R} \to \mathbb{R}$, the linearization of f at $x=a$ is

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

($f(x) \approx L(x)$ for x-values near $x=a$)



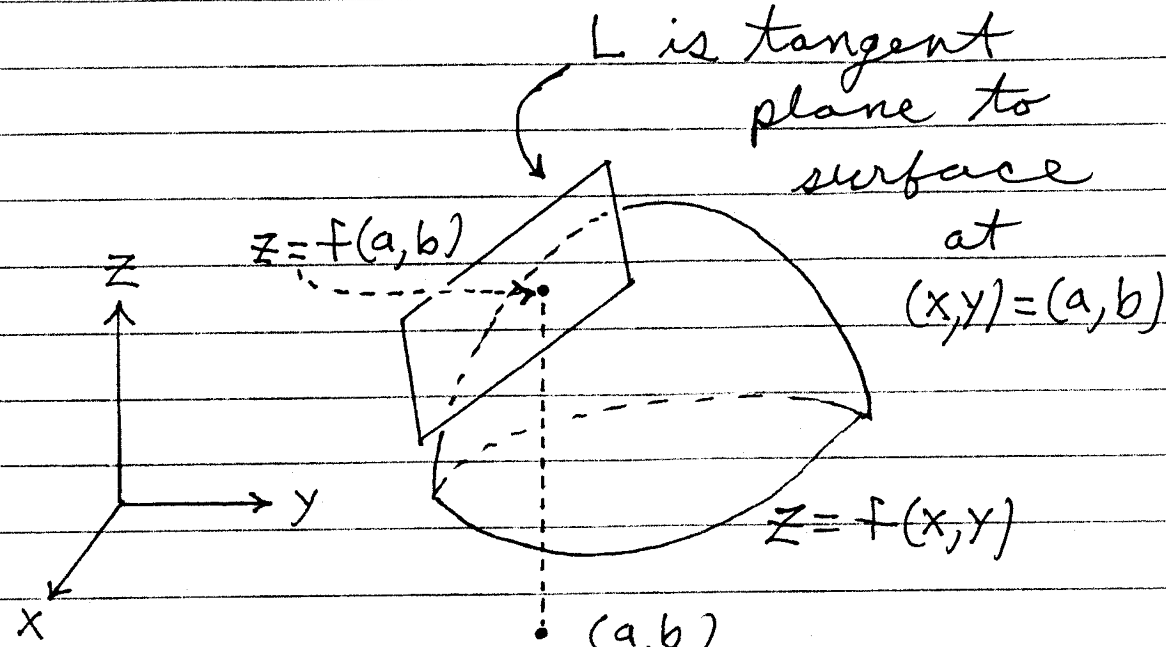
2.) (from Math 17C) For function $z=f(x,y)$ with $f: \mathbb{R}^2 \to \mathbb{R}$, the linearization of f at $(x,y)=(a,b)$ is

$$\boxed{L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

or (matrix form)

$$\boxed{L(x,y) = f(a,b) + \begin{bmatrix} f_x(a,b) & f_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}}$$

$(f(x,y) \approx L(x,y)$ for points (x,y) near (a,b).$)$



Consider a new type of function $f: \mathbb{R}^2 \to \mathbb{R}^2$.

<u>Ex</u>: Let $f(x,y) = \begin{bmatrix} x^2 - 5y \\ x + y^2 \end{bmatrix}$; then

$$f(0,0) = \begin{bmatrix} 0 - 0 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad f(2,1) = \begin{bmatrix} 4 - 5 \\ 2 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \ldots$$

<u>Def</u>: Let function $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$. The <u>Jacobi Matrix</u> (Derivative Matrix) is the 2×2 matrix

$$D\,f(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

__Def__: Let function $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$. The __linearization of f at $(x,y) = (a,b)$__ is

$$L(x,y) = f(a,b) + D\,f(a,b) \cdot \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

or

$$\boxed{L(x,y) = \begin{bmatrix} f_1(a,b) \\ f_2(a,b) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x}(a,b) & \frac{\partial f_1}{\partial y}(a,b) \\ \frac{\partial f_2}{\partial x}(a,b) & \frac{\partial f_2}{\partial y}(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}}.$$

__Fact__: If $L(x,y)$ is the linearization of function $f(x,y)$, then $f(x,y) \approx L(x,y)$ for points (x,y) near (a,b).