Math 17C
Kouba
Linearization and the Jacobi Matrix

Recall: 1.) (from Math 17A) For function \( y = f(x) \) with \( f: \mathbb{R} \rightarrow \mathbb{R} \), the linearization of \( f \) at \( x = a \) is

\[
L(x) = f(a) + f'(a)(x-a)
\]

(\( f(x) \approx L(x) \) for \( x \)-values near \( x = a \))

2.) (from Math 17C) For function \( z = f(x,y) \) with \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \), the linearization of \( f \) at \((x,y) = (a,b)\) is

\[
L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)
\]
or (matrix form)

\[
L(x,y) = f(a,b) + \begin{bmatrix} f_x(a,b) & f_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}
\]
Consider a new type of function $f : \mathbb{R}^2 \to \mathbb{R}^2$.  

**Ex:** Let $f(x, y) = \begin{bmatrix} x^2 - 5y \\ x + y^2 \end{bmatrix}$; then

$f(0, 0) = \begin{bmatrix} 0 - 0 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $f(3, 1) = \begin{bmatrix} 4 - 5 \\ 2 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, ...

**Def:** Let function $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$. The **Jacobian Matrix** (Derivative Matrix) is the $2 \times 2$ matrix
\[ D \ f(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \]

**Def:** Let function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be given

\[ \frac{\partial f}{\partial x} (x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} \quad \text{The linearization} \quad \frac{\partial f}{\partial x} (a, b) \]

\[ L(x, y) = f(a, b) + D f(a, b) \cdot \begin{bmatrix} x-a \\ y-b \end{bmatrix} \]

or

\[ L(x, y) = \begin{bmatrix} f_1(a, b) \\ f_2(a, b) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1(a, b)}{\partial x} & \frac{\partial f_1(a, b)}{\partial y} \\ \frac{\partial f_2(a, b)}{\partial x} & \frac{\partial f_2(a, b)}{\partial y} \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} \]

**Fact:** If \( L(x, y) \) is the linearization of function \( f(x, y) \), then \( f(x, y) \approx L(x, y) \) for points \((x, y)\) near \((a, b)\).