Math 17C
Kouba
Replacing Nonlinear Systems of D.E.'s with Linear Systems

assume $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $f: \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x_1, x_2) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$, and $\frac{dX}{dt} = f(X)$ is a nonlinear 2X2 system of D.E.'s. Let $\hat{X} = (a, b)$ be an equilibrium, so that $f(\hat{X}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and consider $X = \hat{X} + Z$ where $Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, a new variable vector with Z near $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Let $Df(\hat{X})$ be the 2X2 Jacobi Matrix for f evaluated at $X = \hat{X} = (a, b)$.

If $L(X) = L(x_1, x_2)$ is the linearization of f at $X = \hat{X} = (a, b)$, then

$$L(X) = f(\hat{X}) + Df(\hat{X})(X - \hat{X}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + Df(\hat{X}) Z \to$$

$$\boxed{L(X) = Df(\hat{X}) Z}$$

. Note also that

$$\frac{dX}{dt} = \frac{d}{dt}(\hat{X} + Z) = \frac{d}{dt}(\hat{X}) + \frac{dZ}{dt} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{dZ}{dt} \to$$

$$\boxed{\frac{dX}{dt} = \frac{dZ}{dt}}$$

. Since $f(X) \approx L(X)$ for Z near $\hat{X}$, it follows that

$$\frac{dX}{dt} = f(X) \approx L(X) = Df(\hat{X})\,Z \rightarrow$$

$$\frac{dZ}{dt} = \frac{dX}{dt} = Df(\hat{X})\,Z \text{, i.e.,}$$

(II) $\boxed{\dfrac{dZ}{dt} = Df(\hat{X})\,Z}$, a linear 2X2 system of D.E.'s using Z ;

(I) $\boxed{\dfrac{dX}{dt} = f(X)}$, a nonlinear 2X2 system of D.E.'s using X .

The solutions to these systems are related by $\underline{X = \hat{X} + Z}$.

The Hartman - Grobman Theorem states that the direction fields for systems (I) and (II), although different, are very similar for X near the equilibrium $\hat{X}$ for (I) and for Z near $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in (II).

<u>Process</u> : We will use the 2x2 matrix $Df(\hat{X})$ in (II) to determine the stability of the equilibrium $\hat{X}$ in (I) .