Math 17C Kouba

Solving a 2X2 Linear System of D.E.'s

Ex: Solve $X' = \begin{bmatrix} 4 & 1 \\ 6 & 5 \end{bmatrix} X$ and $x_1(0) = 1$, $x_2(0) = -7$:

$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 1 \\ 6 & 5-\lambda \end{bmatrix}$

$= (4-\lambda)(5-\lambda) - (1)(6)$

$= 20 - 4\lambda - 5\lambda + \lambda^2 - 6$

$= \lambda^2 - 9\lambda + 14 = (\lambda-2)(\lambda-7) = 0 \rightarrow$

$\lambda_1 = 2$ or $\lambda_2 = 7$;

Solve $(A - \lambda I)X = 0$ for X:

For $\lambda_1 = 2$: $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 6 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$

$2x_1 + x_2 = 0 \rightarrow x_2 = -2x_1$, let $x_1 = t$ any #

so $x_2 = -2t$ and

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ -2t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ so eigenvector

is $V_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$;

For $\lambda_2 = 7$: $\left[\begin{array}{cc|c} -3 & 1 & 0 \\ 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$

$-3x_1 + x_2 = 0 \rightarrow x_2 = 3x_1$, let $x_1 = t$ any #

so $x_2 = 3t$ and

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 3t \end{bmatrix} = t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ so eigenvector

is $V_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$; the solution is

$$\boxed{X = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{7t}}$$

(vector form)

$$\rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ -2c_1 e^{2t} \end{bmatrix} + \begin{bmatrix} c_2 e^{7t} \\ 3c_2 e^{7t} \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{2t} + c_2 e^{7t} \\ -2c_1 e^{2t} + 3c_2 e^{7t} \end{bmatrix}$$

$$\rightarrow \boxed{\begin{cases} x_1 = c_1 e^{2t} + c_2 e^{7t} \\ x_2 = -2c_1 e^{2t} + 3c_2 e^{7t} \end{cases}}$$

(parametric form) ;

Now find c_1 and c_2 using $x_1(0) = 1$ and $x_2(0) = -7$:

$$\begin{cases} 1 = c_1 e^0 + c_2 e^0 = c_1 + c_2 \\ -7 = -2c_1 e^0 + 3c_2 = -2c_1 + 3c_2 \end{cases}$$

$\rightarrow c_2 = 1 - c_1 \rightarrow$ (SUB) $\rightarrow$

$-7 = -2c_1 + 3(1 - c_1) = 3 - 5c_1 \rightarrow$

$5c_1 = 10 \rightarrow \boxed{c_1 = 2}$ and $c_2 = 1 - (2)$

$\rightarrow \boxed{c_2 = -1} \rightarrow$

$$\boxed{\begin{cases} x_1 = 2e^{2t} - e^{7t} \\ x_2 = 2e^{2t} - 3e^{7t} \end{cases}}$$