Math 17C
Kouba
The Plausibility of the
Eigenvector/Exponential Solution
to a 2x2 Linear System of D.E.'s

Recall: (from Math 17B) If $\frac{dX}{dt} = 3X$ and $X(0) = 4$, then using separation of variables the solution is $X = 4e^{3t}$.

In general, the solution to $\frac{dX}{dt} = aX$ is $X = Ce^{at}$. This leads us to "guess" that the solution to the system of D.E.'s

$$X' = AX$$

takes the form

$$(*) \qquad X = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{\lambda t} ,$$

where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is an unknown vector and λ is an unknown constant. Substituting $(*)$ into $X' = AX$ gives

$$X' = \frac{d}{dt}\left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot e^{\lambda t} \right\} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \lambda e^{\lambda t} \quad \text{or}$$

$$X' = \lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot e^{\lambda t} \quad ;$$

and

$$AX = A\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot e^{\lambda t}\right) = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot e^{\lambda t}.$$

Thus, $X' = AX \longrightarrow$

$$A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot e^{\lambda t} = \lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot e^{\lambda t}$$

(Divide both sides by $e^{\lambda t}$) $\longrightarrow$

$$A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \longrightarrow$$

λ is eigenvalue for A and $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is an associated eigenvector.

How to solve $X' = AX$:

1.) Solve $\det(A - \lambda I) = 0$ to get distinct eigenvalues λ_1, λ_2.

2.) Solve $(A - \lambda I)X = 0$ for X to get distinct eigenvectors V_1, V_2.

3.) The general solution in matrix form is

$$X = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}.$$