Theorem (Second Derivative Test for $f(x,y)$):
Assume $f$ has continuous first and second order partial derivatives and assume $(a,b)$ is a critical point for $f$. Let

$$D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$ 

1. If $D > 0$ and $f_{xx}(a,b) > 0$, then $(a,b)$ determines a relative minimum value at $(a,b, f(a,b))$.

2. If $D > 0$ and $f_{xx}(a,b) < 0$, then $(a,b)$ determines a relative maximum value at $(a,b, f(a,b))$.

3. If $D < 0$, then $(a,b)$ determines a saddle point at $(a,b, f(a,b))$.

4. If $D = 0$, then this test is inconclusive.

Proof:

Let $(h,k)$ be such that $(a+h, b+k)$ is near $(a,b)$ and $(a+th, b+tk)$ is some point between...
(a, b) and (a + h, b + k), i.e., 0 ≤ t ≤ 1 so that
(a + th, b + tk) is on the line segment joining
(a, b) and (a + h, b + k).

Define a new function \( G \) given by
\[ G(t) = f(a + th, b + tk) \]
for 0 ≤ t ≤ 1. By Taylor's formula applied
to \( G(t) \) on \([0, 1]\) we have that

(1) \[ G(1) = G(0) + G'(0) + \frac{1}{2} G''(c) \]

where 0 < c < 1. By the chain rule it follows that

(2) \[ G'(t) = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} = h \cdot f_x + k \cdot f_y \]
and
\[ G''(t) = \frac{d}{dt} (h f_x + k f_y) \cdot \frac{dx}{dt} + \frac{d}{dt} (h f_x + k f_y) \cdot \frac{dy}{dt} \]
\[ = (h f_{xx} + k f_{yx}) h + (h f_{xy} + k f_{yy}) k \]

(3) \[ = h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}, \]

where all partial derivatives are evaluated
at the point \((a + th, b + tk)\). Then by (2)
\[ G'(0) = h f_x(a, b) + k f_y(a, b) = h \cdot 0 + k \cdot 0 = 0 \]
since \((a, b)\) is a critical point for \( f \), and by (3)
\[ G''(c) = A h^2 + 2B h k + C k^2, \]

where \( A = f_{xx}(a+ch,b+ck), \) \( B = f_{xy}(a+ch,b+ck), \) and \( C = f_{yy}(a+ch,b+ck). \) Since \( G(0) = f(a,b) \) and \( G(1) = f(a+h,b+k) \) it follows from (1) that

\[ f(a+h,b+k) = f(a,b) + \frac{1}{2} (A h^2 + 2B h k + C k^2). \]

Let \( k \) be fixed and consider the quadratic expression

\[ q(h) = q(h,k) = Ah^2 + 2B h k + C k^2 = (A)h^2 + (2B)k h + (C k^2). \]

By the quadratic formula the roots of \( q \) are

\[ h = \frac{-2B \pm \sqrt{4B^2 k^2 - 4ACk^2}}{2A} = \frac{-2B \pm \sqrt{B^2 - 4ACk^2}}{2A}. \]

(i) If \( AC - B^2 < 0 \) and \( A > 0, \) then the quadratic in (5) is always positive-valued. It follows from (4) that \( f(a+h,b+k) > f(a,b), \) i.e., \( f(a,b) \) is a minimum value.

(ii) If \( AC - B^2 > 0 \) and \( A < 0, \) then the quadratic in (5) is always negative-valued. It follows from (4) that \( f(a+h,b+k) < f(a,b), \) i.e., \( f(a,b) \) is a
(iii) If $AC - B^2 < 0$, then the quadratic in (5) assumes both positive and negative values. It follows from (4) that $f(a+h,b+k) < f(a,b)$ for some $(h,k)$ and $f(a+h,b+k) > f(a,b)$ for some $(h,k)$, i.e., $f$ has a saddle point at $(a,b)$.

By the continuity of $f$ and its partial derivatives, it follows that

$$D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

and

$$AC - B^2$$

have the same sign, and $f_{xx}(a,b)$ and $A$ have the same sign. Thus, 1.), 2.), and 3.) follow from (i), (ii), and (iii).

Q.E.D.