Math 17C
Kouba
Establish a Shortcut for Getting Information About Eigenvalues for a $2X2$ Matrix
Establish a Shortcut for Establishing the Stability of Equilibrium $(0,0)$

<u>*DEFINITION*</u> : Let matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

1.) The <u>*determinant*</u> of matrix A is $\det(A) = ad - bc$.

2.) The <u>*trace*</u> of matrix A is $\mathrm{tr}(A) = a + d$ (the sum of the main diagonal numbers).

<u>*NOTE*</u> : Since matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\det(A - \lambda I) = \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - (b)(c)$$

$$= \lambda^2 - a\lambda - d\lambda + ad - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad - bc) = 0 \quad \text{(equation (*))}$$

We also know that if λ_1 and λ_2 are eigenvalues for matrix A, then

$$\det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

$$\longrightarrow \quad \lambda^2 - \lambda_1\lambda - \lambda_2\lambda + \lambda_1\lambda_2 = 0$$

$$\longrightarrow \quad \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0 \quad \text{(equation (**))}$$

Combining equations (*) and (**) leads to the following EIGENVALUE SHORTCUTS :

<u>*FACT*</u> : Consider matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its eigenvalues λ_1 and λ_2.

1.) $\mathrm{tr}(A) = a + d = \lambda_1 + \lambda_2$ (SUM of eigenvalues)

2.) $\det(A) = ad - bc = \lambda_1\lambda_2$ (PRODUCT of eigenvalues)

Determine the $\mathrm{tr}(A)$ and $\det(A)$ for each of the following cases for λ_1 and λ_2 :

<u>*case 1*</u> : If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $\mathrm{tr}(A) = \lambda_1 + \lambda_2 > 0$ and $\det(A) = \lambda_1\lambda_2 > 0$

(and equilibrium $(0,0)$ is unstable, source)

<u>*case 2*</u> : If $\lambda_1 < 0$ and $\lambda_2 < 0$, then $\mathrm{tr}(A) = \lambda_1 + \lambda_2 < 0$ and $\det(A) = \lambda_1\lambda_2 > 0$

(and equilibrium $(0,0)$ is STABLE, sink)

<u>case 3</u> : If $\lambda_1 < 0$ and $\lambda_2 > 0$, then $\text{tr}(A) = \lambda_1 + \lambda_2$ is INCONCLUSIVE (why ?) and $\det(A) = \lambda_1\lambda_2 < 0$

(and equilibrium $(0,0)$ is unstable, saddle)

<u>case 4</u> : If $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ with $a > 0$, then $\text{tr}(A) = \lambda_1 + \lambda_2 = 2a > 0$ and $\det(A) = \lambda_1\lambda_2 = a^2 + b^2 > 0$

(and equilibrium $(0,0)$ is unstable, unstable spiral)

<u>case 5</u> : If $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ with $a < 0$, then $\text{tr}(A) = \lambda_1 + \lambda_2 = 2a < 0$ and $\det(A) = \lambda_1\lambda_2 = a^2 + b^2 > 0$

(and equilibrium $(0,0)$ is STABLE, stable spiral)

<u>case 6</u> : If $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ with $a = 0$, then $\text{tr}(A) = \lambda_1 + \lambda_2 = 0$ and $\det(A) = \lambda_1\lambda_2 = b^2 > 0$

(and equilibrium $(0,0)$ is unstable, neutral spiral)

These six cases explain the following result :

<u>*THEOREM*</u> : Let A be a 2X2 matrix with eigenvalues λ_1 and λ_2. Then the real parts of λ_1 and λ_2 are both negative iff

$$\text{tr}(A) < 0 \quad \text{AND} \quad \det(A) > 0$$

and equilibrium $(0,0)$ is STABLE. If either condition fails to be true, then equilibrium $(0,0)$ is UNSTABLE.