Math 17C (Winter 2016)
Kouba
Exam 1

KEY

Please PRINT your name here : --

Your Exam ID Number ------------

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. COPYING ANSWERS FROM ANOTHER STUDENT'S EXAM IS A VIOLATION OF THE UNIVERSITY HONOR CODE. HAVING ANOTHER STUDENT TAKE YOUR EXAM FOR YOU IS A VIOLATION OF THE UNIVERSITY HONOR CODE. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. No notes, books, or classmates may be used as resources for this exam.

5. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

6. You have until 3 p.m. sharp to finish the exam. PLEASE STOP WRITING IMMEDIATELY when time is called and close your exam. Failure to close your exam in a timely fashion could result in points being deducted from your exam score.

7. Make sure that you have 9 pages including the cover page.

1.) Consider the function given by $6x - 3y + 2z = 12$ and its graph in 3D-space.

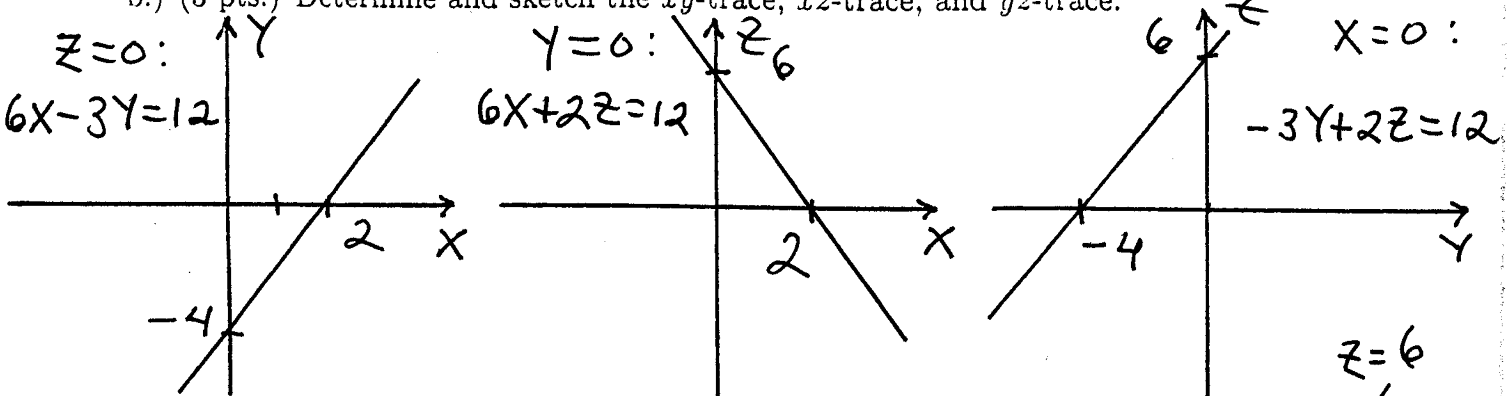
a.) (3 pts.) Determine all possible intercepts for this equation.

$X=0,\ Y=0:\quad Z=6$

$X=0,\ Z=0:\quad Y=-4$

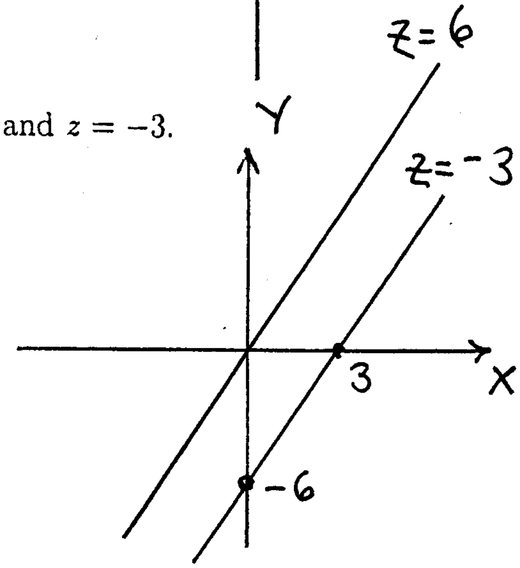
$Y=0,\ Z=0:\quad X=2$

b.) (3 pts.) Determine and sketch the xy-trace, xz-trace, and yz-trace.



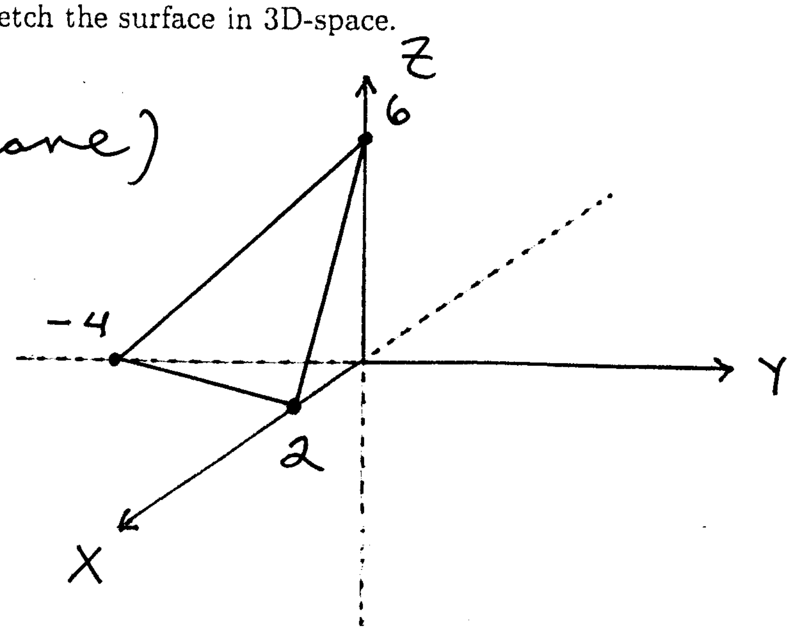
c.) (2 pts.) Sketch level curves on the same axes for $z = 6$ and $z = -3$.

z-value	level curve
6	$6X-3Y+12=12$
	$\rightarrow 6X-3Y=0$
	$\rightarrow Y=2X$
−3	$6X-3Y-6=12$
	$\rightarrow 6X-3Y=18$
	$\rightarrow 2X-Y=6$



d.) (4 pts.) Sketch the surface in 3D-space.

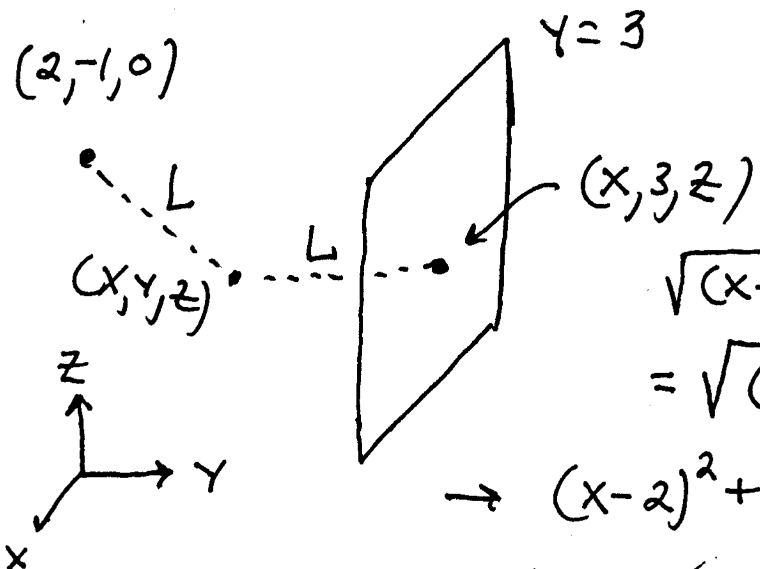
(plane)



2.) (8 pts.) Let $z = x \cdot \sin(2x+3y)$. Compute the partial derivatives z_x, z_y, and z_{xy} . SIMPLIFY your answers.

$$z_X = X \cdot \cos(2X+3Y) \cdot 2 + (1)\sin(2X+3Y)$$
$$= 2X\cos(2X+3Y) + \sin(2X+3Y),$$
$$z_Y = X \cdot \cos(2X+3Y) \cdot 3 = 3X\cos(2X+3Y),$$
$$z_{XY} = 2X \cdot -\sin(2X+3Y) \cdot 3$$
$$+ \cos(2X+3Y) \cdot 3$$
$$= -6X\sin(2X+3Y) + 3\cos(2X+3Y)$$

3.) (6 pts.) Determine an equation for the set of all points (x, y, z) which are equidistant from the point $(2, -1, 0)$ and the plane $y = 3$. Simplify your answer as much ass possible.



$L = L \rightarrow$

$$\sqrt{(X-2)^2 + (Y+1)^2 + (z-0)^2}$$
$$= \sqrt{(X-X)^2 + (Y-3)^2 + (z-z)^2}$$
$$\rightarrow (X-2)^2 + (Y+1)^2 + z^2 = (Y-3)^2$$
$$\rightarrow (X-2)^2 + \cancel{Y^2} + 2Y + 1 + z^2 = \cancel{Y^2} - 6Y + 9$$
$$\rightarrow \boxed{(X-2)^2 + z^2 = 8 - 8Y}$$

OR $X^2 - 4X + 4 + z^2 = 8 - 8Y$

$$\rightarrow X^2 + z^2 = 4X - 8Y + 4$$

4.) (8 pts.) Evaluate the following limit : $\displaystyle\lim_{(x,y)\to(3,-2)} \frac{xy-3y-2x+6}{9-x^2}$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{(x,y)\to(3,-2)} \frac{y(x-3)-2(x-3)}{(3-x)(3+x)}$$

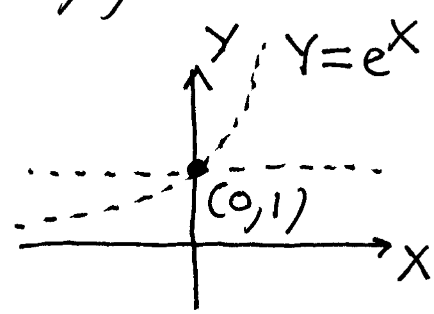
$$= \lim_{(x,y)\to(3,-2)} \frac{\cancel{(x-3)}(y-2)}{-\cancel{(x-3)}(3+x)}$$

$$= \frac{-2-2}{-(3+3)} = \frac{-4}{-6} = \frac{2}{3}$$

5.) (8 pts.) Verify that the following limit does not exist : $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x+\ln y}{xy} = \text{"}\frac{0}{0}\text{"};$

<u>Path $y=1$</u> : $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x+\overset{0}{\cancel{\ln 1}}}{x(1)} = \lim_{(x,y)\to(0,1)} \frac{x}{x}$

$$= \lim_{(x,y)\to(0,1)} 1 = 1$$



<u>Path $y=e^x$</u> : $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x+\ln e^x}{xe^x}$

$$= \lim_{(x,y)\to(0,1)} \frac{x+x}{xe^x} = \lim_{(x,y)\to(0,1)} \frac{2}{e^x} = \frac{2}{e^0} = 2$$

and $1 \neq 2$ so $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x+\ln y}{xy}$ DNE

6.) a.) (5 pts.) Determine and sketch the domain of $f(x,y) = \sqrt{25 - x^2 - y^2} + \sqrt{x^2 + y^2 - 9}$ in 2D-space.

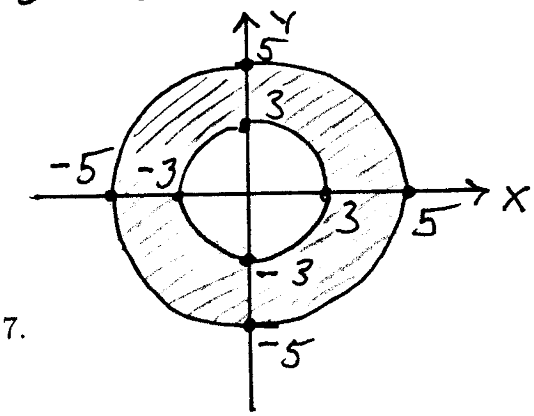
Domain is set of all points (x, y) on or outside circle $x^2 + y^2 = 3^2$ and on or inside circle $x^2 + y^2 = 5^2$

$25 - x^2 - y^2 \geq 0$

$\rightarrow x^2 + y^2 \leq 5^2$ AND

$x^2 + y^2 - 9 \geq 0$

$\rightarrow x^2 + y^2 \geq 3^2$



b.) (5 pts.) Determine the range of $f(x,y) = x^2 + 3y^2 - 7$.

$0 \leq x^2 + 3y^2 < \infty$ so

$-7 \leq \underbrace{x^2 + 3y^2 - 7}_{z} < \infty$

Range: all $z \geq -7$

7.) (8 pts.) Give a precise ϵ, δ-proof that $\lim_{(x,y)\to(1,-1)} (x - y^2) = 0$.

Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < \sqrt{(x-1)^2 + (y+1)^2} < \delta$, then $|x - y^2 - 0| < \varepsilon$. Begin with $|x - y^2|$ and solve for $\sqrt{(x-1)^2 + (y+1)^2}$. Then

$|x - y^2| = |(x-1) + 1 - (y+1)^2 + 2y + 1|$

$= |(x-1) - (y+1)^2 + 2(y+1)|$

$\leq |x-1| + (y+1)^2 + 2|y+1|$

$= \sqrt{(x-1)^2} + (y+1)^2 + 2\sqrt{(y+1)^2}$

$\leq \sqrt{(x-1)^2 + (y+1)^2} + (x-1)^2 + (y+1)^2 + 2\sqrt{(x-1)^2 + (y+1)^2}$

$= 3\sqrt{(x-1)^2 + (y+1)^2} + \left(\sqrt{(x-1)^2 + (y+1)^2}\right)^2$ (assume $\delta \leq 1$)

$\leq 3\sqrt{(x-1)^2 + (y+1)^2} + \sqrt{(x-1)^2 + (y+1)^2}$

$= 4\sqrt{(x-1)^2 + (y+1)^2} < \varepsilon$

iff $\sqrt{(x-1)^2 + (y+1)^2} < \varepsilon/4$.

Choose $\delta = \min\{\frac{\varepsilon}{4}, 1\}$.

Done.

Q.E.D.

8.) (8 pts.) Consider the surface given by $z = x^3 + y^2$ and point $P = (-1, 2, 3)$ on this surface. Determine an equation for the plane tangent to this surface at point P.

$f_x = 3x^2$, $f_y = 2y$ so tangent plane is

$z = f(-1,2) + f_x(-1,2)(x-(-1)) + f_y(-1,2)(y-2) \rightarrow$

$z = 3 + 3(x+1) + 4(y-2)$ or

$z = 3x + 4y - 2$.

9.) (8 pts.) Consider the function $f(x, y) = (1/2)x^2 - y$ and the point $P = (-1, 1)$. Find all unit vectors $\vec{u}$ so that the directional derivative of $D_{\vec{u}} f(-1, 1) = 0$.

Let $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ with $\boxed{a^2 + b^2 = 1}$; gradient

$\nabla f(x,y) = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} x \\ -1 \end{bmatrix}$, $\nabla f(-1,1) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and

$D_{\vec{u}} f(-1,1) = \nabla f(-1,1) \cdot \vec{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = -a - b = 0$

$\rightarrow \boxed{a = -b}$; then $(-b)^2 + b^2 = 1 \rightarrow 2b^2 = 1$

$\rightarrow b^2 = \frac{1}{2} \rightarrow b = \pm \frac{1}{\sqrt{2}}$; so $b = \frac{1}{\sqrt{2}}$, $a = \frac{-1}{\sqrt{2}}$ OR

$b = \frac{-1}{\sqrt{2}}$, $a = \frac{1}{\sqrt{2}}$, so

$\vec{u} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ or $\vec{u} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

10.) (8 pts.) Find the linearization $L(x,y)$ of $f(x,y) = \begin{pmatrix} x - e^y \\ xy \end{pmatrix}$ at the point $(2,0)$. Simplify your answer as much as possible.

$$Df(x,y) = \begin{bmatrix} 1 & -e^y \\ y & x \end{bmatrix} \rightarrow Df(2,0) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix},$$

$f(2,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so linearization is

$$L(x,y) = f(2,0) + Df(2,0) \cdot \begin{bmatrix} x-2 \\ y-0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x-2 \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} x-2-y \\ 2y \end{bmatrix} \rightarrow$$

$$L(x,y) = \begin{bmatrix} x-1-y \\ 2y \end{bmatrix}$$

11.) (6 pts.) The temperature T of a flat plate at point (x,y) is given by $T = x^2 - y^3$. If you are standing at point (2,1), in which direction $\vec{u}$ should you move in order to get to a hotter point as fast as possible and at what rate does the temperature increase ? Write your direction as a unit vector.

$D_{\vec{u}} f(2,1)$ is largest in the direction of the gradient vector:

$$\vec{\nabla} f(x,y) = \begin{bmatrix} 2x \\ -3y^2 \end{bmatrix} \rightarrow \vec{\nabla} f(2,1) = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \rightarrow$$

$$\vec{u} = \frac{\vec{\nabla} f(2,1)}{|\vec{\nabla} f(2,1)|} = \frac{\begin{bmatrix} 4 \\ -3 \end{bmatrix}}{\sqrt{4^2 + (-3)^2}} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$$

and rate of temp. change is

$$D_{\vec{u}} f(2,1) = |\vec{\nabla} f(2,1)| = 5.$$

12.) (10 pts.) Find and classify (relative maximum value, relative minimum value, or saddle point) each critical point for the following function :

$z = 3x^2 + 2y^3 - 6xy$

$\to z_X = 6X - 6Y = 0 \to \boxed{Y = X}$,

$\to z_Y = 6Y^2 - 6X = 0 \to \boxed{X = Y^2} \to$

(sub) $\to Y = Y^2 \to 0 = Y^2 - Y = Y(Y-1)$

$\to Y=0, X=0$ or $Y=1, X=0 \to$

crit. pts. are $\boxed{(0,0)}$ and $\boxed{(1,1)}$;

$\to z_{XX} = 6$, $z_{YY} = 12Y$, $z_{XY} = -6$

$\to D = (6)(12Y) - (-6)^2$

For $(0,0)$: $D = (6)(0) - 36 = -36 \to$

$(0,0)$ det. a SADDLE POINT at $z=0$.

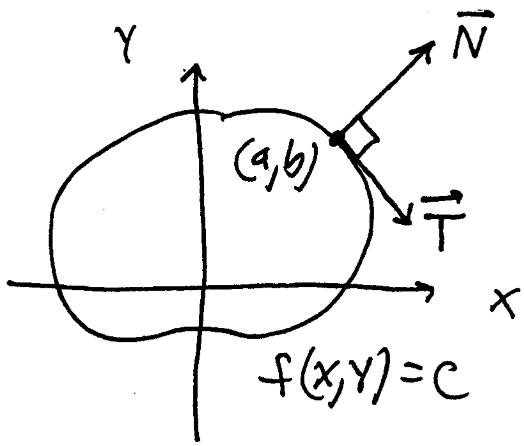
For $(1,1)$: $D = (6)(12) - 36 = 36 > 0$ and

$z_{XX} = 6 > 0 \to (0,1)$ det. a MIN at

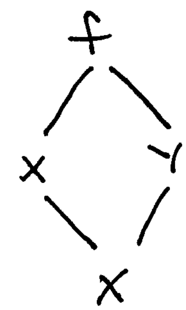
$z = 2$.

The following EXTRA CREDIT PROBLEM is worth 8 points. This problem is OPTIONAL.

1.) Consider the function $z = f(x, y)$ and the point $(x, y) = (a, b)$. Explain (prove) why the gradient vector $\nabla f(a, b)$ is perpendicular to the level curve $f(x, y) = C$, where $C = f(a, b)$.



apply chain rule to $f(x,Y)=C$, $Y=g(x)$:



$$f_X \cdot \underbrace{\frac{dx}{dx}}_{1} + f_Y \cdot \frac{dY}{dx} = 0$$

$\rightarrow \quad \dfrac{dY}{dx} = \dfrac{-f_X}{f_Y} = \dfrac{\Delta Y}{\Delta X}$, so tangent vector at point (a,b) can be $\vec{T} = \begin{bmatrix} f_Y \\ -f_X \end{bmatrix}$;

the gradient vector at (a,b) is

$\nabla f = \begin{bmatrix} f_X \\ f_Y \end{bmatrix}$; since $\vec{T} \cdot \nabla f = (f_Y)(f_X) + (-f_X)(f_Y) = 0$,

∇f is $\perp$ to $\vec{T}$, i.e.,

∇f is $\perp$ to level curve $f(x,Y) = C$.