Math 17C (Winter 2016)
Kouba
Exam 2

Please PRINT your name here : ____KEY____

Your Exam ID Number ______

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

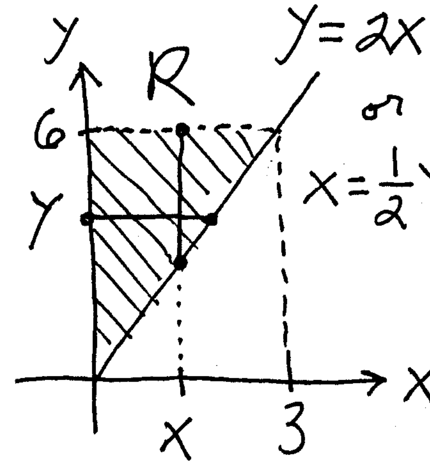
5. Make sure that you have 10 pages, including the cover page.

6. You will be graded on proper use of limit, derivative, and integral notation.

7. You have until 3:02 p.m. sharp to finish the exam. Students who fail to stop working when time is called at the end of the exam MAY HAVE POINTS DEDUCTED from their score.

1.) (4 pts. each) Consider region R bounded by the graphs of $y = 2x$, $y = 6$, and $x = 0$. Describe R using

a.) vertical cross-sections.



$$\begin{cases} 0 \le X \le 3 \\ 2X \le Y \le 6 \end{cases}$$

b.) horizontal cross-sections.

$$\begin{cases} 0 \le Y \le 6 \\ 0 \le X \le \frac{1}{2}Y \end{cases}$$

2.) (5 pts.) The point $(0,0)$ is an equilibrium for the following system of differential equations. Determine if $(0,0)$ is an unstable or stable equilibrium. Then categorize $(0,0)$ as a sink, source, saddle, stable spiral, unstable spiral, or neutral spiral.

$X' = \begin{pmatrix} -2 & 3 \\ 0 & 5 \end{pmatrix} X$

$$\det(A - \lambda I) = \det\begin{bmatrix} -2-\lambda & 3 \\ 0 & 5-\lambda \end{bmatrix}$$

$$= (-2-\lambda)(5-\lambda) - (3)(0)$$

$$= (-2-\lambda)(5-\lambda) = 0 \to \lambda_1 = -2, \ \lambda_2 = 5$$

so $(0,0)$ is UNSTABLE (SADDLE)

3.) (10 pts.) Solve the following system of differential equations with initial conditions. Write your answer in matrix (vector) form.

$$X' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} X \quad \text{and} \quad x_1(0) = 0,\ x_2(0) = -1$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 0-\lambda & 2 \\ -2 & 0-\lambda \end{bmatrix}$$

$$= (-\lambda)(-\lambda) - (2)(-2) = \lambda^2 + 4 = 0 \rightarrow \lambda^2 = -4$$

$\rightarrow \lambda = \pm 2i$; choose $\lambda = 2i$:

Solve $A - \lambda I X = 0$: $\left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & i & 0 \end{array}\right]$

$\rightarrow x_1 + i x_2 = 0$, so let $x_2 = t$ (any #) $\rightarrow$

$x_1 = -i x_2 = -it$, so $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$

so eigenvector is $V = \begin{bmatrix} -i \\ 1 \end{bmatrix}$; then solution is

$$X = c_1 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{2it} = c_1 \begin{bmatrix} -i \\ 1 \end{bmatrix} (\cos 2t + i \sin 2t)$$

$$= c_1 \begin{bmatrix} -i \cos 2t - \overset{-1}{i^2} \sin 2t \\ \cos 2t + i \sin 2t \end{bmatrix}$$

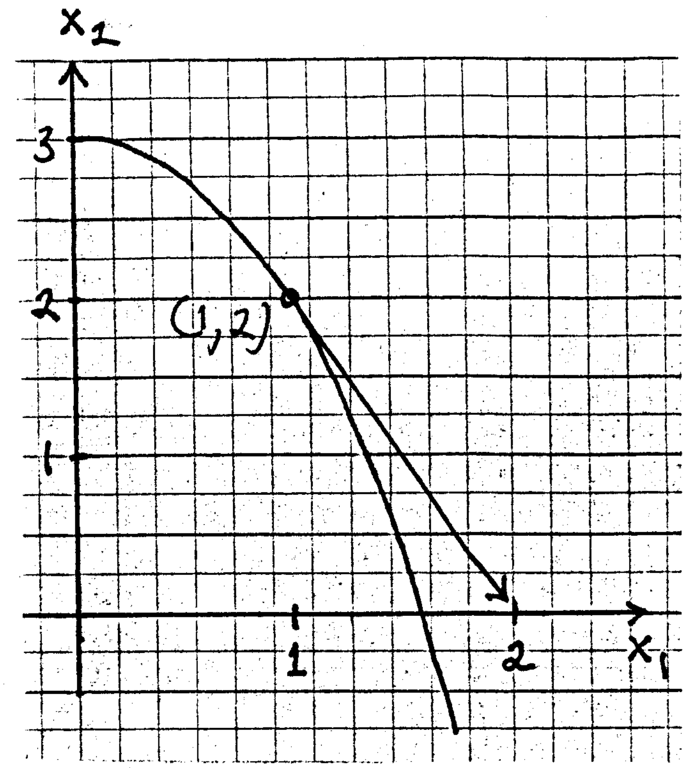
$$= c_1 \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} + c_1 i \begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix}$$

$$= c_1 \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix}$$

4.) The position (x_1, x_2) of a particle at time t is given parametrically by

$$\begin{cases} x_1 = e^t \\ x_2 = 3 - e^{2t} \quad \text{for } -\infty < t < \infty \end{cases}$$

and its graph is given below. For $t = 0$ determine



a.) (1 pt.) and plot the point (x_1, x_2).

$x_1 = e^0 = 1, \quad x_2 = 3 - e^0 = 2$

b.) (2 pts.) the slope of the graph.

$$SLOPE = \frac{x_2'}{x_1'} = \frac{-2e^{2t}}{e^t} = \frac{-2e^0}{e^0} = -2$$

c.) (3 pts.) and sketch a direction vector.

$$X' = \begin{bmatrix} e^t \\ -2e^{2t} \end{bmatrix} = \begin{bmatrix} e^0 \\ -2e^0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

d.) (3 pts.) the speed of the particle.

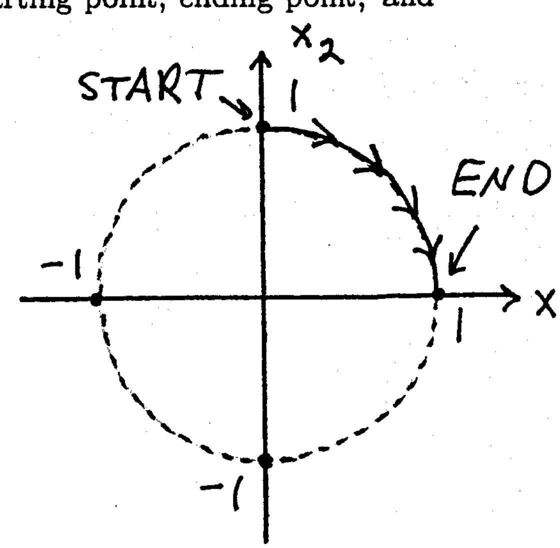
Speed

$$\frac{ds}{dt} = \sqrt{(x_1')^2 + (x_2')^2}$$

$$= \sqrt{(1)^2 + (-2)^2} = \sqrt{5}$$

5.) (8 pts.) The position (x_1, x_2) of a particle at time t is given parametrically by the following. Eliminate t and write the path as an equation in only x_1 and x_2. Then sketch the graph of the path in the x_1x_2-plane, indicating the starting point, ending point, and direction of motion of the particle.

$$\begin{cases} x_1 = \sin 2t \\ x_2 = \cos 2t, \quad \text{for } 0 \le t \le \pi/4 \end{cases}$$



$$x_1^2 + x_2^2 = (\sin 2t)^2 + (\cos 2t)^2$$

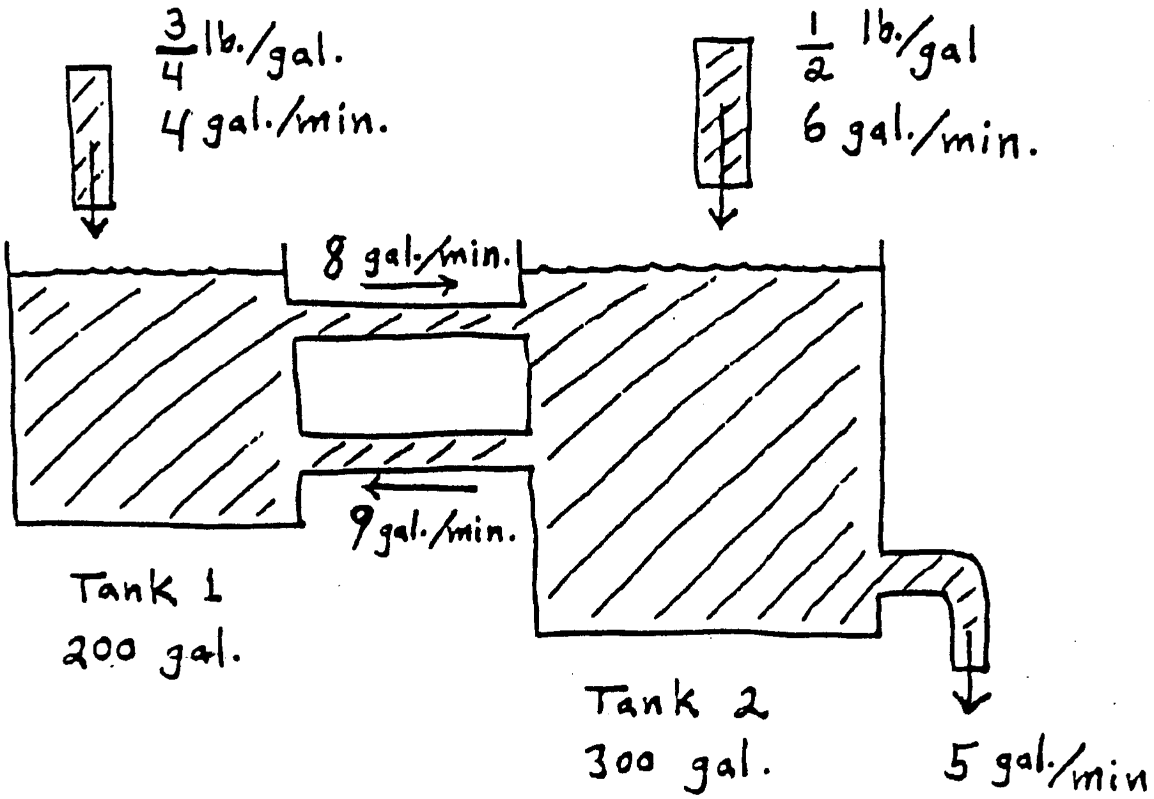
$$\rightarrow x_1^2 + x_2^2 = 1 \quad \text{(circle)}$$

$t = 0: \quad (0, 1)$

$t = \frac{\pi}{8}: \quad \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$t = \frac{\pi}{4}: \quad (1, 0)$

6.) (8 pts.) Consider the two tanks containing water and salt mixtures and connected as shown in the diagram. Tank 1 initially holds 200 gallons of mixture. Tank 2 initially holds 300 gallons of mixture. Let x_1 and x_2 represent the pounds of salt in Tank 1 and Tank 2, resp., at time t. Initially, Tank 1 contains 40 pounds of salt and Tank 2 contains 25 pounds of salt. The mixture in each tank is kept uniform by stirring, and the mixtures are pumped from each tank to the other at the rates indicated in the figure. In addition, a mixture containing 3/4 pound of salt per gallon is pumped into Tank 1 at 4 gal./min.; a mixture containing 1/2 pound of salt per gallon is pumped into Tank 2 at 6 gal./min.; the mixture leaves Tank 2 at 5 gal./min. SET UP, BUT DO NOT SOLVE, a sytem of differential equations with initial conditions, which represents the amount of salt in each tank. PAY CLOSE ATTENTION TO FLOW RATES IN AND OUT OF EACH TANK !!!!



$$\frac{dx_1}{dt} = \left(\frac{3}{4}\ \frac{\text{lb.}}{\text{gal.}}\right)\left(4\ \frac{\text{gal}}{\text{min}}\right) + \left(\frac{x_2\ \text{lbs.}}{300\,\text{gal}}\right)\left(9\ \frac{\text{gal.}}{\text{min.}}\right) - \left(\frac{x_1\ \text{lbs.}}{200+5t\ \text{gal.}}\right)\left(8\ \frac{\text{gal.}}{\text{min.}}\right)$$

$$\frac{dx_2}{dt} = \left(\frac{1}{2}\ \frac{\text{lb.}}{\text{gal.}}\right)\left(6\ \frac{\text{gal.}}{\text{min.}}\right) + \left(\frac{x_1\ \text{lbs.}}{200+5t\ \text{gal.}}\right)\left(8\ \frac{\text{gal.}}{\text{min.}}\right)$$

$$- \left(\frac{x_2\ \text{lbs.}}{300\ \text{gal.}}\right)\left(9\ \frac{\text{gal.}}{\text{min.}}\right) - \left(\frac{x_2\ \text{lbs.}}{300\ \text{gal.}}\right)\left(5\ \frac{\text{gal.}}{\text{min.}}\right)$$

$x_1(0) = 40$ lbs., $x_2(0) = 25$ lbs.

7.) (10 pts.) (Drug Dissipation Model) Morphine is a potent opiate analgesic drug that is used to relieve severe pain. Assume that 25 mg. of morphine is administered to an injured athlete in a single dose. Let x_1 and x_2 be the mg. of morphine in the person's body tissue and urinary tract, resp., at time t hours. Assume that the amount of morphine in the body tissue dissipates exponentially and that the urinary tract contains 10 mg. of morphine after 4 hours. Set up a system of differential equations with initial conditions and solve this system for x_1 and x_2.

$$\begin{cases} \dfrac{dx_1}{dt} = -kx_1 , & x_1(0) = 25 \text{ mg.} \\ \dfrac{dx_2}{dt} = +kx_1 , & x_2(0) = 0 \text{ mg.} \end{cases}$$

and $x_2(4) = 10$ mg. , $x_1 + x_2 = 25$;

$$\rightarrow x_1 = Ce^{-kt} = 25e^{-kt} \quad \text{and}$$

$$x_2 = 25 - 25e^{-kt} , \quad x_2(4) = 10 \rightarrow$$

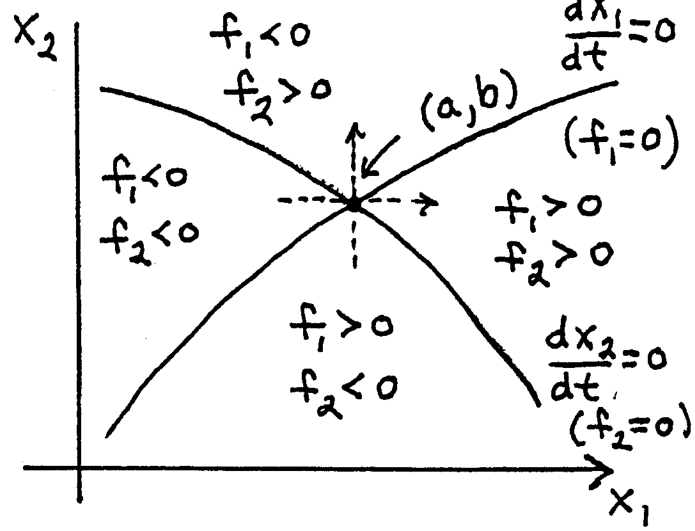
$$10 = 25 - 25e^{-4k} \rightarrow 25e^{-4k} = 15 \rightarrow$$

$$e^{-4k} = \frac{15}{25} = \frac{3}{5} \rightarrow \ln e^{-4k} = \ln(3/5) \rightarrow$$

$$-4k = \ln(3/5) \rightarrow k = \frac{-1}{4}\ln(3/5) \quad \text{so}$$

$$\begin{cases} x_1 = 25e^{\frac{1}{4}\ln(3/5)\cdot t} \\ x_2 = 25 - 25e^{\frac{1}{4}\ln(3/5)\cdot t} \end{cases}$$

8.) (Graphical Approach) Consider the given graph of zero isoclines with equilibrium (a, b) and with a sign chart for f_1 and f_2. Use this sign chart to create a signed Jacobi Matrix. Then determine if the equilibrium is stable or unstable or if this method is inconclusive.



(10 pts.) a.) Along + x_1-arrow:

$f_1: (-) \to (+)$, so $\frac{\partial f_1}{\partial x_1}$ is $(+)$

$f_2: (-) \to (+)$, so $\frac{\partial f_2}{\partial x_1}$ is $(+)$;

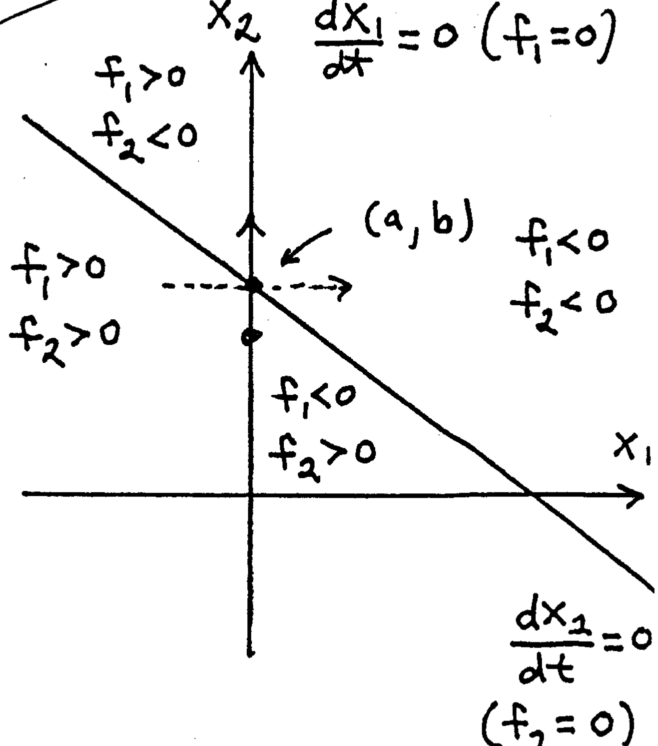
along + x_2-arrow:

$f_1: (+) \to (-)$ so $\frac{\partial f_1}{\partial x_2}$ is $(-)$

$f_2: (-) \to (+)$ so $\frac{\partial f_2}{\partial x_2}$ is $(+)$,

$$Df(a,b) = \begin{bmatrix} + & - \\ + & + \end{bmatrix};$$ so $\operatorname{tr} A = (+) + (+) = (+)$, $\det A = (+)(+) - (-)(+) = (+)$,

so (a,b) is UNSTABLE



(5 pts.) b.) Along + x_1-arrow:

$f_1: (+) \to (-)$, so $\frac{\partial f_1}{\partial x_1}$ is $(-)$

$f_2: (+) \to (-)$, so $\frac{\partial f_2}{\partial x_1}$ is $(-)$;

along + x_2-arrow:

$f_1: (0) \to (0)$, so $\frac{\partial f_1}{\partial x_2} = 0$

$f_2: (+) \to (-)$, so $\frac{\partial f_2}{\partial x_2}$ is $(-)$;

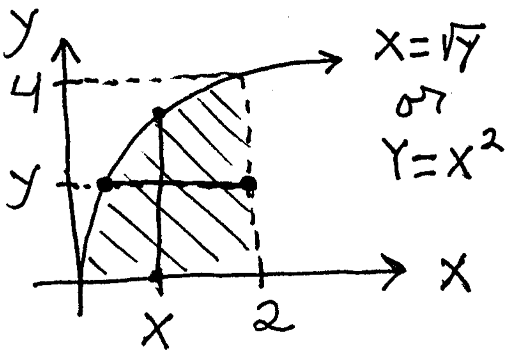
$$Df(a,b) = \begin{bmatrix} - & 0 \\ - & - \end{bmatrix};$$ so $\operatorname{tr} A = (-) + (-) = (-)$, $\det A = (-)(-) - (0)(-) = (+)$,

so (a,b) is STABLE

9.) (8 pts.) Evaluate the given double integral. (HINT: First REVERSE the ORDER of integration.)

a.) $\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy$

$$\begin{cases} 0 \le Y \le 4 \\ \sqrt{Y} \le X \le 2 \end{cases}$$



$$= \int_0^2 \int_0^{X^2} e^{X^3}\, dy\, dX \qquad \begin{cases} 0 \le X \le 2 \\ 0 \le Y \le X^2 \end{cases}$$

$$= \int_0^2 \left(e^{X^3} Y \Big|_{Y=0}^{Y=X^2} \right) dX$$

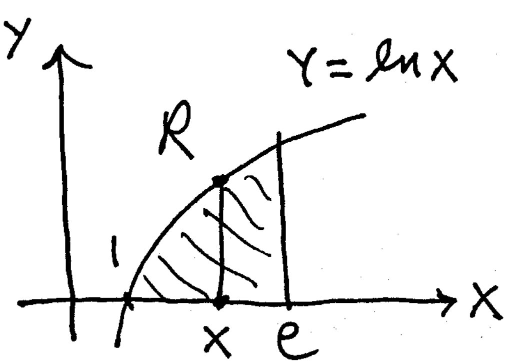
$$= \int_0^2 X^2 e^{X^3}\, dX = \frac{1}{3} e^{X^3} \Big|_0^2$$

$$= \frac{1}{3} e^8 - \frac{1}{3} e^0 = \frac{1}{3} e^8 - \frac{1}{3}$$

10.) (8 pts.) Consider the region R bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$. SET UP BUT DO NOT EVALUATE INTEGRAL(S) for the *average value* of function $f(x, y) = xy^2$ on region R.

$$\text{Area } R = \int_1^e \int_0^{\ln x} 1\, dy\, dx$$

then



$$\text{AVE} = \frac{1}{\text{Area } R} \int_1^e \int_0^{\ln x} x Y^2\, dy\, dx$$

11.) Consider the following nonlinear system of differential equations.

$$\frac{dx_1}{dt} = 2x_1 - x_2^2$$
$$\frac{dx_2}{dt} = x_2 - x_1$$

a.) (4 pts.) Determine all zero isoclines for this system.

$2x_1 - x_2^2 = 0 \rightarrow \boxed{2x_1 = x_2^2}$

OR $\quad x_2 - x_1 = 0 \rightarrow \boxed{x_2 = x_1}$ (SUB)

b.) (4 pts.) Determine all equilibria for this system.

$2x_1 = x_1^2 \rightarrow 0 = x_1^2 - 2x_1 = x_1(x_1 - 2) = 0$

$\rightarrow \boxed{x_1 = 0, x_2 = 0}$ OR $\boxed{x_1 = 2, x_2 = 2}$

c.) (8 pts.) For each equilibrium use the Analytical Approach (Jacobi Matrix and eigenvalues) to determine stability and classify it, or state that this method is inconclusive.

$$Df(x_1,x_2) = \begin{bmatrix} 2 & -2x_2 \\ -1 & 1 \end{bmatrix} \rightarrow$$

For $(0,0)$: $Df(0,0) = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \rightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 \\ -1 & 1-\lambda \end{bmatrix} = 0 \rightarrow$

$(2-\lambda)(1-\lambda) - (0)(-1) = 0 \rightarrow \lambda_1 = 1, \lambda_2 = 2$ so

$(0,0)$ UNSTABLE (SOURCE)

For $(2,2)$: $Df(2,2) = \begin{bmatrix} 2 & -4 \\ -1 & 1 \end{bmatrix} \rightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & -4 \\ -1 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) - (-4)(-1)$

$= \lambda^2 - \lambda - 2\lambda + 2 - 4 = \lambda^2 - 3\lambda - 2 = 0 \rightarrow$

$$\lambda = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2} = \frac{3 \pm \sqrt{17}}{2} \rightarrow \begin{array}{l} \lambda_1 = \frac{1}{2}(3+\sqrt{17}) > \\ \lambda_2 = \frac{1}{2}(3-\sqrt{17}) < 0 \end{array}$$

so $(2,2)$ UNSTABLE (SADDLE)

The following EXTRA CREDIT PROBLEM is worth 8 points. This problem is OPTIONAL.

Determine a linear $2x2$ sytem of differential equations for which the following two equations are solutions :

$$\begin{cases} x_1 = e^t + e^{-t} \\ x_2 = 2e^t + 3e^{-t}, \qquad \text{for } -\infty < t < \infty \end{cases}$$

Linear: $\begin{cases} \dfrac{dx_1}{dt} = ax_1 + bx_2 \\ \dfrac{dx_2}{dt} = cx_1 + dx_2 \end{cases} \rightarrow$ (SUB)

$$\frac{dx_1}{dt} = e^t - e^{-t} = a(e^t + e^{-t}) + b(2e^t + 3e^{-t}) \rightarrow$$

$$(1)e^t + (-1)e^{-t} = (a+2b)e^t + (a+3b)e^{-t} \rightarrow$$

$$\begin{cases} a+2b=1 \\ a+3b=-1 \end{cases} \rightarrow b=-2, \quad a = 1-2b = 5 \quad ;$$

$$\frac{dx_2}{dt} = 2e^t - 3e^{-t} = c(e^t + e^{-t}) + d(2e^t + 3e^{-t}) \rightarrow$$

$$(2)e^t + (-3)e^{-t} = (c+2d)e^t + (c+3d)e^{-t} \rightarrow$$

$$\begin{cases} c+2d=2 \\ c+3d=-3 \end{cases} \rightarrow d=-5, \ c = 2-2d = 12, \text{ so}$$

$$\begin{cases} \dfrac{dx_1}{dt} = 5x_1 - 2x_2 \\ \dfrac{dx_2}{dt} = 12x_1 - 5x_2 \end{cases}$$