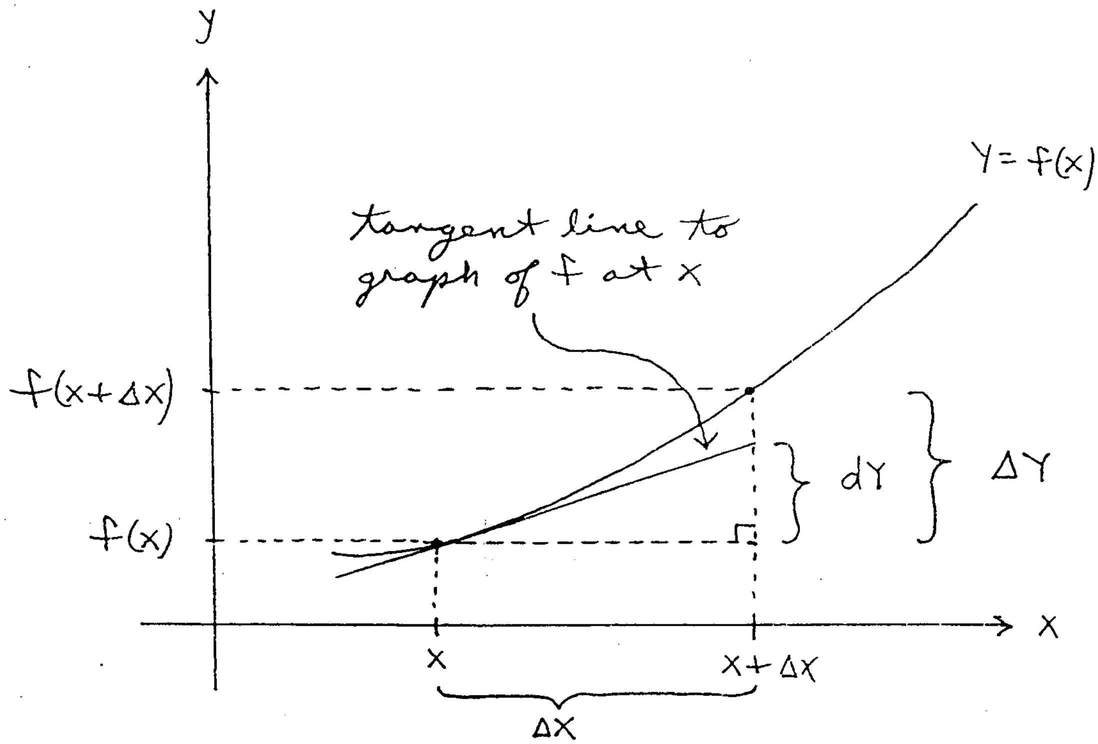
Math 21A
Koúba
The Differential



Define the exact change in y to be

$$\boxed{\Delta Y = f(x+\Delta x) - f(x)} \, .$$

Define dY to be the height of the right triangle in the diagram. Then

$$\boxed{\Delta Y \approx dY} \quad \text{for} \quad \Delta x \quad \text{small} .$$

In addition, the slope of the tangent line to the graph of f at x is

$$f'(x) = \frac{\text{rise}}{\text{run}} = \frac{dY}{\Delta x} \qquad \text{so that}$$

$$\boxed{dY = f'(x) \cdot \Delta x} \; ;$$

dY is called the differential of Y.

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More Examples Using Differentials

Example 1: For small h, show that $\sqrt{4+3h^2} \approx 2 + \frac{3}{4}h^2$ using differentials.

Solution: Let $f(x) = \sqrt{x}$ and assume that $x: 4 \to 4+3h^2$. Then $\Delta x = 3h^2$ and $f'(x) = \frac{1}{2\sqrt{x}}$. Since Δx is small (because h is small)

$$\Delta f \approx df \to f(4+3h^2) - f(4) \approx f'(4) \cdot \Delta x$$
$$\to \sqrt{4+3h^2} - \sqrt{4} \approx \frac{1}{2\sqrt{4}} \cdot 3h^2$$
$$\to \sqrt{4+3h^2} - 2 \approx \frac{3}{4}h^2$$
$$\to \sqrt{4+3h^2} \approx 2 + \frac{3}{4}h^2 .$$

Example 2: If the radius of a circle is measured with an absolute percentage error of at most 3%, use differentials to estimate the maximum absolute percentage error in computing the circle's a.) circumference. b.) area.

Solution: assume that $\frac{|\Delta r|}{r} \le 3\%$.

a.) $C = 2\pi r$, $C' = 2\pi$, find $\frac{|\Delta C|}{C}$:

$$\frac{|\Delta C|}{C} \approx \frac{|dC|}{C} = \frac{|C' \cdot \Delta r|}{C} = \frac{|2\pi \cdot \Delta r|}{2\pi r} = \frac{|\Delta r|}{r} \le 3\%$$

b.) $A = \pi r^2$, $A' = 2\pi r$, find $\frac{|\Delta A|}{A}$:

$$\frac{|\Delta A|}{A} \approx \frac{|dA|}{A} = \frac{|A' \cdot \Delta r|}{A} = \frac{|2\pi r \cdot \Delta r|}{\pi r^2} = 2\,\frac{|\Delta r|}{r} \le 2(3\%)$$
$$= 6\%$$

The Conversion of Mass to Energy

Here is an example of how the approximation

$$\frac{1}{\sqrt{1-x^2}} \approx 1 + \frac{1}{2}x^2 \tag{4}$$

is used in an applied problem.

Newton's second law,

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma,$$

is stated with the assumption that mass is constant, but we know this is not strictly true because the mass of a body increases with velocity. In Einstein's corrected formula, mass has the value

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \tag{5}$$

where the "rest mass" m_0 represents the mass of a body that is not moving and c is the speed of light, which is about 300,000 km/sec. When v is very small compared with c, v^2/c^2 is close to zero and it is safe to use the approximation

$$\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)$$

(Eq. 4 with $x = v/c$) to write

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0\left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)\right] = m_0 + \frac{1}{2}m_0v^2\left(\frac{1}{c^2}\right),$$

or

$$m \approx m_0 + \frac{1}{2}m_0v^2\left(\frac{1}{c^2}\right). \tag{6}$$

Equation (6) expresses the increase in mass that results from the added velocity v.

In Newtonian physics, $(1/2)m_0v^2$ is the kinetic energy (KE) of the body, and if we rewrite Eq. (6) in the form

$$(m - m_0)c^2 \approx \frac{1}{2}m_0v^2,$$

we see that

$$(m - m_0)c^2 \approx \frac{1}{2}m_0v^2 = \frac{1}{2}m_0v^2 - \frac{1}{2}m_0(0)^2 = \Delta(\text{KE}),$$

or

$$(\Delta m)c^2 \approx \Delta(\text{KE}). \tag{7}$$

In other words, the change in kinetic energy $\Delta(\text{KE})$ in going from velocity 0 to velocity v is approximately equal to $(\Delta m)c^2$.

With c equal to 3×10^8 m/sec, Eq. (7) becomes

$$\Delta(\text{KE}) \approx 90{,}000{,}000{,}000{,}000{,}000\ \Delta m \text{ joules}$$ mass in kilograms

and we see that a small change in mass can create a large change in energy. The energy released by exploding a 20-kiloton atomic bomb, for instance, is the result of converting only 1 gram of mass to energy. The products of the explosion weigh only 1 gram less than the material exploded. A U.S. penny weighs about 3 grams.