Math 21A
Kouba
Proving the Derivative of Logarithm

Let $f(x) = \log_b(x)$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\log_b(x+h) - \log_b(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \log_b \left(\frac{x+h}{x} \right) \qquad \text{(by property of logarithms)}$$

$$= \lim_{h \to 0} \frac{1}{h} \log_b (1 + h/x)$$

$$= \lim_{h \to 0} \log_b (1 + h/x)^{1/h} \qquad \text{(by property of logarithms)}$$

$$= \lim_{h \to 0} \log_b \left[(1 + h/x)^{1/(h/x)} \right]^{(1/x)}$$

$$= \log_b [e]^{(1/x)} \qquad \text{(by definition of } e\text{)}$$

$$= \frac{1}{x} \log_b [e] \qquad \text{(by property of logarithms)}$$

If $b = 10$, then $\quad D \log x = \dfrac{1}{x} \log e$.

If $b = e$, then $\quad D \ln x = \dfrac{1}{x}$.