Math 21A
Kouba
Precise Definition of Limit

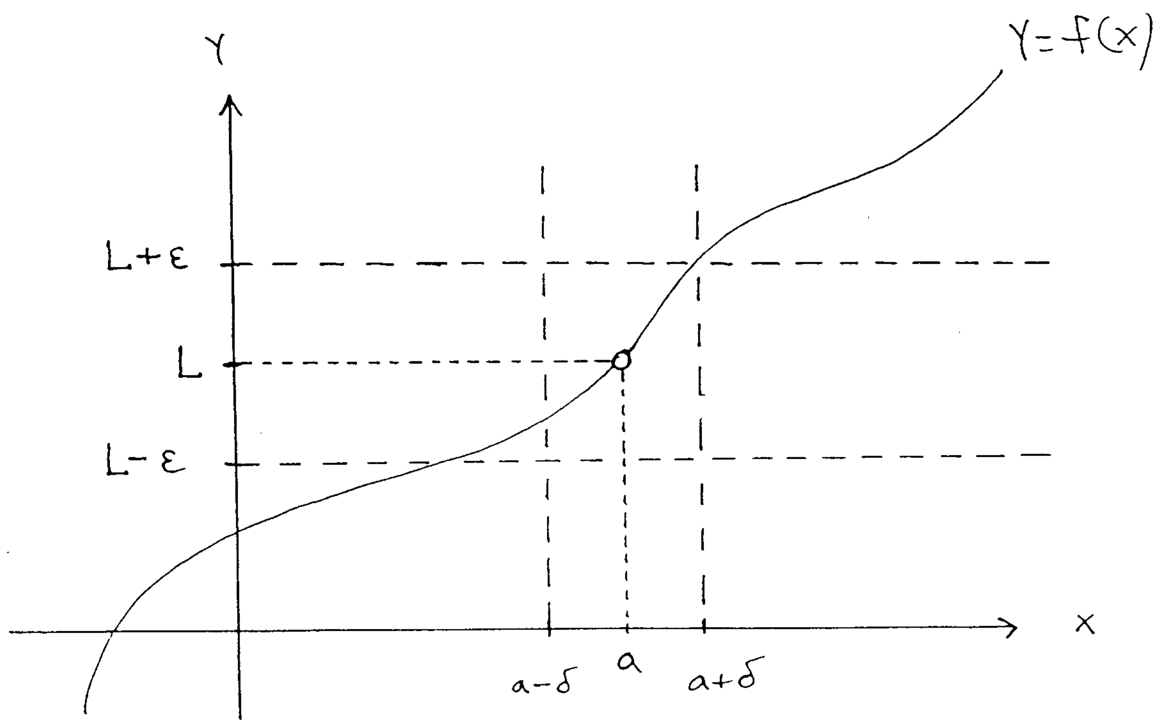
__Def__: The statement

$$\lim_{x \to a} f(x) = L$$

has the following precise definition :
Given any number $\varepsilon > 0$, there exists another number $\delta > 0$ so that

if $\quad 0 < |x-a| < \delta$

then $\quad |f(x) - L| < \varepsilon$.



Math 21A
Kouba
ε, δ- Proof

<u>Example</u>: Use the precise definition of limit to prove that $\lim_{x \to -5} \frac{x+1}{3-x} = \frac{-1}{2}$:

Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < |x-(-5)| < \delta$, then $|f(x) - (\frac{-1}{2})| < \varepsilon$, i.e., if $0 < |x+5| < \delta$, then $|f(x) + \frac{1}{2}| < \varepsilon$.

Begin with $|f(x)+\frac{1}{2}| < \varepsilon$ and "solve for" $|x+5|$. Then

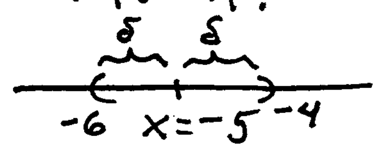
$$|f(x)+\tfrac{1}{2}| < \varepsilon \text{ iff } \left|\frac{x+1}{3-x} + \frac{1}{2}\right| < \varepsilon$$

$$\text{iff } \left|\frac{2(x+1)+(3-x)}{2(3-x)}\right| < \varepsilon$$

$$\text{iff } \frac{1}{2}\left|\frac{x+5}{3-x}\right| < \varepsilon$$

$$\text{iff } \frac{|x+5|}{|3-x|} < 2\varepsilon .$$

At this point we need to "eliminate" the term $|3-x|$. Assume that $\delta \leq 1$ so that $-6 < x < -4$ and $7 < |3-x| < 9$ and $\frac{1}{9} < \frac{1}{|3-x|} < \frac{1}{7}$. Then



$$\frac{|x+5|}{|3-x|} < \frac{1}{7}|x+5| < 2\varepsilon \text{ iff } \frac{1}{7}|x+5| < 2\varepsilon$$

$$\text{iff } |x+5| < 14\varepsilon .$$

Now choose $\delta = \min\{1, 14\varepsilon\}$.

Thus, if $0 < |x+5| < \delta$, then $|f(x)+\frac{1}{2}| < \varepsilon$.

This completes the proof. $\blacksquare$