Section 2.3

34:4 \[ \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = 6 \]

34:5 \[ \lim_{x \to 1} \frac{x^4 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x^2 - 1)(x^2 + 1)}{(x-1)(x^2 + x + 1)} = \lim_{x \to 1} \frac{(x-1)(x+1)(x^2 + 1)}{(x-1)(x^2 + x + 1)} = \frac{(2)(2)}{3} = \frac{4}{3} \]

34:8 \[ \lim_{x \to 5} \frac{3x + 5}{4x} = \frac{15 + 5}{20} = 1 \]

34:10 \[ \pi^2 = \pi^2 \]

34:13 \[ \lim_{x \to 1^+} \frac{x - 1}{|x - 1|} = \lim_{x \to 1^+} \frac{x - 1}{x - 1} = 1 \]

(See graph) \[ = \lim_{x \to 1^+} 1 = 1 \]

34:14 \[ \lim_{x \to 1^-} \frac{x - 1}{|x - 1|} = \lim_{x \to 1^-} \frac{x - 1}{-(x - 1)} = \lim_{x \to 1^-} -1 = -1 \]

34:15 \[ \lim_{h \to 1} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 1} \frac{4 - 1}{h} = 3 \]

34:16 \[ \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h \to 0} \frac{2h}{h} = 2 \]

34:17 \[ \lim_{x \to 2} \frac{1}{x} = \frac{1}{2} = \lim_{x \to 2} \frac{2-x}{2x} \cdot \frac{1}{x-2} \]

= \lim_{x \to 2} \frac{-1}{2x} = -\frac{1}{4}
\[ \lim_{x \to 1} \frac{3^x - 3}{2^x} = \frac{0}{2} = 0 \]

(a) \( \lim_{x \to 1} f(x) = 2 \)
(b) \( \lim_{x \to 2} f(x) = 2 \)
(c) \( \lim_{x \to 3} f(x) = 1 \)
(d) \( \lim_{x \to 4^-} f(x) = 2 \)

\[
\begin{array}{ccc}
X & \frac{3^x - 1}{x} \\
0.1 & 1.1612317 \\
0.01 & 1.1046692 \\
0.001 & 1.0992159 \\
0.0001 & 1.0986724 \\
-0.0001 & 1.0985514 \\
\end{array}
\]

\[ \lim_{x \to 0} \frac{3^x - 1}{x} \approx 1.098 \]

It will be shown later that the limit is \( \ln 3 \).

\[ \lim_{x \to 3} f(x) = 1 \]

\[ \lim_{x \to 3} f(x) = 0 \]

\[ \lim_{x \to 3.5} f(x) = 0 \]

\[ \lim_{x \to a} f(x) \text{ exists for all values of } a \]

\[ f(x) = \begin{cases} 
  x^2, & x \text{ rational} \\
  x^3, & x \text{ irrational} 
\end{cases} \]
b.) \( \lim_{x \to 2} f(x) \) does not exist

c.) \( \lim_{x \to 1} f(x) = 1 \)
d.) \( \lim_{x \to 0} f(x) = 0 \)
e.) \( \lim_{x \to a} f(x) \) exists for \( a = 0 \) and \( a = 1 \) only.

\[ f(n) = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n} \quad (n+1 \text{ terms}) \]

a.) 
\[ f(1) = 1 + \frac{1}{2} = \frac{3}{2} = 1.5 \]
\[ f(2) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} = 1.0833333 \]
\[ f(3) = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{57}{60} = 0.95 \]
\[ f(4) = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = 0.8845238 \]
\[ f(5) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} = 0.8456349 \]
\[ f(6) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} = 0.8198773 \]
\[ f(7) = \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} = 0.8015623 \]
\[ f(8) = \frac{1}{8} + \frac{1}{9} + \ldots + \frac{1}{16} = 0.7878718 \]
\[ f(9) = \frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{18} = 0.7772509 \]
\[ f(10) = \frac{1}{10} + \frac{1}{11} + \ldots + \frac{1}{20} = 0.7687744 \]

b) As \( n \) increases, \( f(n) \) decreases.

c.) d.) Since
\[ \frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n} < \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n} < \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} \]
\[ \frac{n+1}{2n} < f(n) < \frac{n+1}{n} \]
\[ \lim_{n \to \infty} \frac{n+1}{2n} \leq \lim_{n \to \infty} f(n) \leq \lim_{n \to \infty} \frac{n+1}{n} \to \]

\[ \lim_{n \to \infty} \left( \frac{1}{2} + \frac{1}{2n} \right) \leq \lim_{n \to \infty} f(n) \leq \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) \to \]

\[ \frac{1}{2} + 0 \leq \lim_{n \to \infty} f(n) \leq 1 + 0 \to \]

\[ \frac{1}{2} \leq \lim_{n \to \infty} f(n) \leq 1 . \]

This conjecture follows from the succession of algebraically equivalent double inequalities.

Remark: It can be shown later that

\[ \lim_{n \to \infty} f(n) = \ln 2 . \]